Facial residual functions and error bounds for cones

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1 Introduction

Consider the following feasibility problem.

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & x \in (\mathcal{L} + a) \cap \mathcal{K} \end{array} \qquad \text{(Feas)}$$

- $\mathcal{K}$: closed convex cone contained in some space $\mathcal{E}$.
- $\mathcal{L}$: subspace contained in $\mathcal{E}$.
- $a \in \mathcal{E}$.

Goal: We want to estimate $\operatorname{dist}(x, (\mathcal{L} + a) \cap \mathcal{K})$ using $\operatorname{dist}(x, \mathcal{L} + a)$ and $\operatorname{dist}(x, \mathcal{K})$.

2 Facial residual functions and error bounds

Proposition 2.1 (An error bound for when a face satisfying a CQ is known). *Let $\mathcal{F} \trianglelefteq \mathcal{K}$ be a face such that*

(a) $\mathcal{F} \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$

(b) $(\operatorname{ri} \mathcal{F}) \cap (\mathcal{L} + a) \neq \emptyset$

Then, for every bounded set B, there exists $\kappa_B > 0$ such that

$$\operatorname{dist}(x, \mathcal{K} \cap (\mathcal{L} + a)) \leq \kappa_B(\operatorname{dist}(x, \mathcal{F}) + \operatorname{dist}(x, \mathcal{L} + a)), \qquad \forall x \in B.$$

Idea:

1. Find $\mathcal{F}$ such that

(a) $\mathcal{F} \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$

(b) $(\operatorname{ri} \mathcal{F}) \cap (\mathcal{L} + a) \neq \emptyset$

Therefore, by Proposition 2.1

$$\operatorname{dist}(x, \mathcal{K} \cap (\mathcal{L} + a)) \leq \kappa_B(\operatorname{dist}(x, \mathcal{F}) + \operatorname{dist}(x, \mathcal{L} + a)), \quad \forall x \in B. \quad (1)$$

2. Upper bound $\operatorname{dist}(x, \mathcal{F})$ using $\operatorname{dist}(x, \mathcal{K})$ and $\operatorname{dist}(x, \mathcal{L} + a)$.

3. Plug the upper bound in (1).

Definition 2.2 ($\mathbb{1}$-FRF for $\mathcal{K}$ and z). A function $\psi_{\mathcal{K},z} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a *one-step facial residual function ($\mathbb{1}$-FRF) for $\mathcal{K}$ and z* if

1. $\psi_{\mathcal{K},z}$ is nonnegative, monotone nondecreasing in each argument and $\psi(0, \alpha) = 0$ for every $\alpha \in \mathbb{R}_+$.
2. for $x \in \operatorname{span} \mathcal{K}$ and $\epsilon \geq 0$ we have

$$\operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \ \langle x, z \rangle \leq \epsilon \quad \Rightarrow \quad \operatorname{dist}(x, \mathcal{K} \cap \{z\}^\perp) \leq \psi_{\mathcal{K},z}(\epsilon, \|x\|).$$

Theorem 2.3 (Error bound based on $\mathbb{1}$-FRFs). *Let $\mathcal{K}$ be a closed convex cone such that $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$. There are $\ell > 0$, $\kappa > 0$, $M > 0$ such that*

$$x \in B, \quad \operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \quad \operatorname{dist}(x, \mathcal{L} + a) \leq \epsilon,$$

implies

$$\operatorname{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa(\epsilon + \varphi(\epsilon, M)),$$

where $\varphi = \psi_{\ell-1} \diamondsuit \cdots \diamondsuit \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let $\varphi(\epsilon, \|x\|) := \epsilon$.

- *ℓ is such that there is chain of faces of $\mathcal{K}$ together with $z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp$ such that*

  $$(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F}_\ell \neq \emptyset.$$

  and $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$ for every i. This is obtained through ***facial reduction*** *[1, 5]*
- *The ψ_i's are $\mathbb{1}$-FRFs for $\mathcal{F}_i$, z_i.*

3 Examples

Definition 3.1 (Hölderian error bound). C_1, C_2 satisfy a **uniform Hölderian error bound** $\overset{\text{def}}{\Longleftrightarrow}$ there exists $\gamma \in (0, 1]$ such that for every bounded set B there exist $\theta_B > 0$, such that

$$\operatorname{dist}(x, C_1 \cap C_2) \leq \theta_B(\operatorname{dist}(x, C_1) + \operatorname{dist}(x, C_2))^\gamma \quad \forall\, x \in B.$$

If $\gamma = 1$, we call it a **Lipschitzian** error bound.

Suppose that (Feas) is feasible.

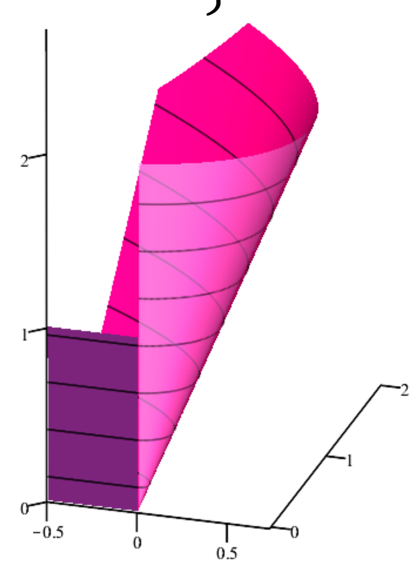
- $\mathcal{K}$: a symmetric cone (psd matrices, second order cone and etc), the $\mathbb{1}$-FRFs are of the form $\psi_{\mathcal{F},z}(\epsilon, t) = \kappa\epsilon + \kappa\sqrt{\epsilon t} \Rightarrow$ (Feas) satisfy an uniform Hölderian error bound with exponent $2^{-\ell}$, see [4]
- p-cone $\mathcal{K}_p^{n+1} := \{x = (x_0, \bar{x}) \in \mathbb{R}^{n+1} \mid x_0 \geq \|\bar{x}\|_p\}$, $\mathbb{1}$-FRF: $\psi_{\mathcal{K}_p^{n+1},z}(\epsilon, t) = \kappa\epsilon + \kappa(\epsilon t)^{\alpha_z}$

$$\alpha_z := \begin{cases} \frac{1}{2} & \text{if } |\bar{z}|_0 = n, \\ \frac{1}{p} & \text{if } |\bar{z}|_0 = 1 \text{ and } p < 2, \\ \min\left\{\frac{1}{2}, \frac{1}{p}\right\} & \text{otherwise}, \end{cases}$$

(Feas) satisfy an uniform Hölderian error bound with exponent α_z, see [3].

3.1 The exponential cone case

$$K_{\exp} := \left\{(x, y, z) \mid y > 0, z \geq y e^{x/y}\right\} \cup \{(x, y, z) \mid x \leq 0, z \geq 0, y = 0\}.$$



Four types of error bounds are possible. Depending on the the vector $\mathbf{z}$, besides Lipschitzian and Hölderian error bound with exponent $1/2$ we have

- **Entropic error bound**: for every bounded set B, there exists $\kappa_B > 0$ such that for $\mathbf{x} \in B$

  $$\operatorname{dist}\left(\mathbf{x}, (\mathcal{L} + a) \cap K_{\exp}\right) \leq \kappa_B \mathfrak{g}_{-\infty}(\max(\operatorname{dist}(\mathbf{x}, \mathcal{L} + a), \operatorname{dist}(\mathbf{x}, K_{\exp}))),$$

- **Logarithmic error bound**: for every bounded set B, there exists $\kappa_B > 0$ such that for $\mathbf{x} \in B$

  $$\operatorname{dist}\left(\mathbf{x}, (\mathcal{L} + a) \cap K_{\exp}\right) \leq \kappa_B \mathfrak{g}_{\infty}(\max(\operatorname{dist}(\mathbf{x}, \mathcal{L} + a), \operatorname{dist}(\mathbf{x}, K_{\exp}))),$$

where $\mathfrak{g}_{-\infty}(t) := -t\ln(t)$, $\mathfrak{g}_\infty(t) := -\frac{1}{\ln(t)}$, (for t small). See [2].

参考文献

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