A comparison of bootstrap methods for causality analyses of multivariate time series

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1 Introduction

There have been proposed many methods of linear connectivity detection. The most popular method is the Granger causality test [1], which is for investigating the flow of information between time series by comparing the variance of the residuals of the Auro-Regressive(AR) model and AR model with eXogenous input (ARX model). One of the other methods is to evaluate the difference in the AIC between the AR and ARX models [2][3]. Another approach is Impulse Response (IR) analysis to quantify causality and feedback mechanisms among variables from Vector AR (VAR) model parameters. Although it is possible to evaluate the significance of the results from the analyses mentioned above using the asymptotic method, the bootstrap method, a more flexible method, is often practical in real data analysis.

There are several bootstrap methods, such as Block Bootstrap (BB), Time-shift Bootstrap (TB), AR-sieve Bootstrap (ARB), and FFT Bootstrap (FB)[4]. So far, it has not been evaluated which combination of causal analysis and which Bootstrap method has what degree of estimation performance, so this study made those evaluations.

2 Causal analyses

AR model
$$x_t = \sum_{i=1}^{\infty} a_{1i} x_{t-i} + \varepsilon_{1t},\quad y_t = \sum_{i=1}^{\infty} b_{1i} y_{t-i} + \varepsilon_{2t}$$

AR Exogenous (ARX) model
$$\begin{cases} x_t = \sum_{i=1}^{\infty} a_{2i} x_{t-i} + \sum_{i=1}^{\infty} c_{2i} y_{t-i} + \varepsilon_{3t} \\ y_t = \sum_{i=1}^{\infty} b_{2i} y_{t-i} + \sum_{i=1}^{\infty} d_{2i} x_{t-i} + \varepsilon_{4t} \end{cases}$$

Granger Causality (GC)
$$F_{y\to x} = \ln\left(\frac{\mathrm{var}(\varepsilon_{1t})}{\mathrm{var}(\varepsilon_{3t})}\right) \qquad F_{x\to y} = \ln\left(\frac{\mathrm{var}(\varepsilon_{2t})}{\mathrm{var}(\varepsilon_{4t})}\right)$$

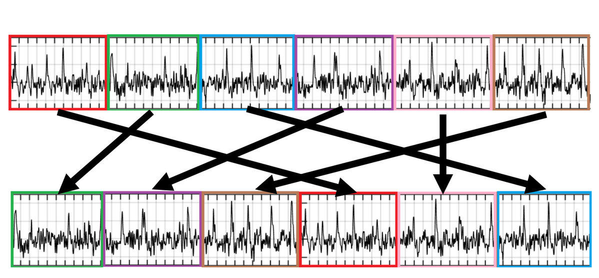
Difference of AIC (dAIC)
$$\Delta\mathrm{AIC}_{y\to x} = \mathrm{AIC}\{\mathrm{ARX}(x_t)\} - \mathrm{AIC}\{\mathrm{AR}(x_t)\}$$
$$\Delta\mathrm{AIC}_{x\to y} = \mathrm{AIC}\{\mathrm{ARX}(y_t)\} - \mathrm{AIC}\{\mathrm{AR}(y_t)\}$$
$$\frac{\Delta\mathrm{AIC}_{y\to x}}{\sqrt{\widehat{\mathrm{var}}(\Delta\mathrm{AIC}_{y\to x})}},\ \frac{\Delta\mathrm{AIC}_{x\to y}}{\sqrt{\widehat{\mathrm{var}}(\Delta\mathrm{AIC}_{x\to y})}} \sim N(0,1)$$

Impulse Response (IR) Analysis
$$Y_t = \sum_{i=1}^{p} \hat{A}_i Y_{t-i} + E_t$$
Vector AR model

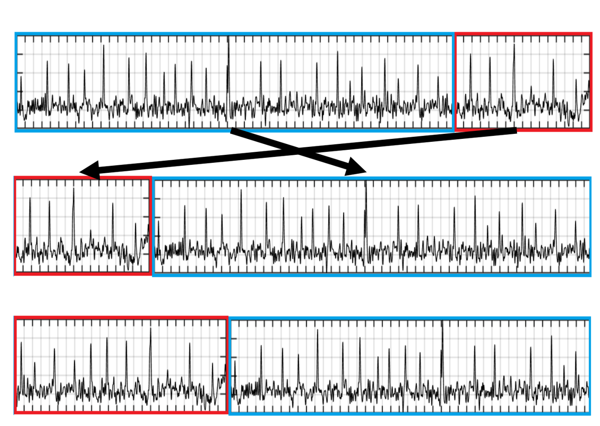
Impact $= e_j$　　j: impacted channeal
e.g. e_j=$(1,0,0,0,0)^t,(0,1,0,0,0)^t$, and so on

3 Boot strap methods

Block Bootstrap (BB)



Time Shift Bootstrap (TB)



About 2000 repetitions

AR-sieve Bootstrap (ARB)

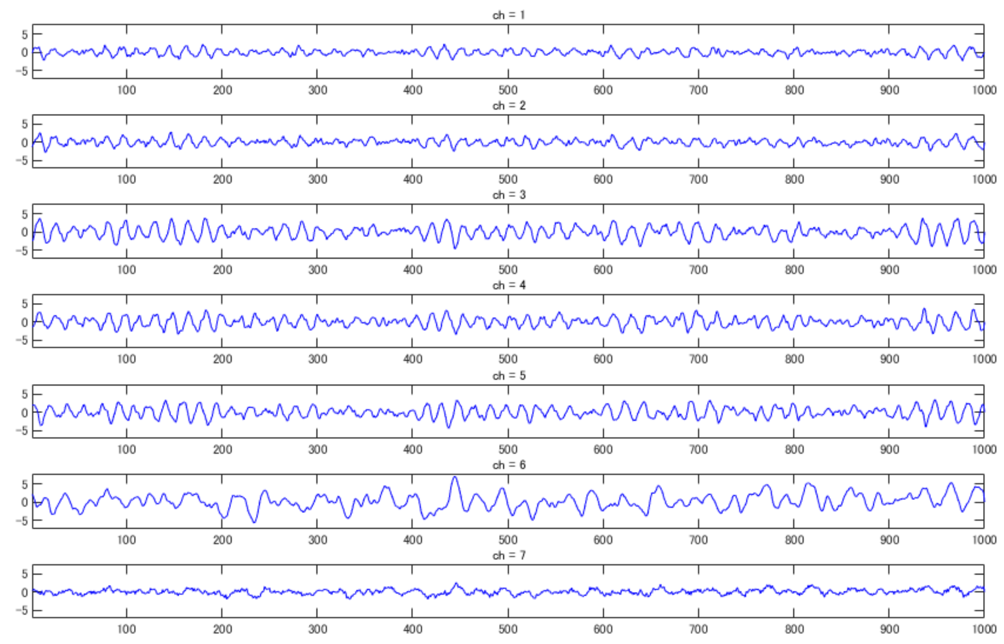
$$Y_t = \sum_{i=1}^{p} \hat{A}_i Y_{t-i} + E_t \ \rightarrow\ \text{Randomly Sampled residuals (with Replacement)}\ \{E_{p+1}^{*b}, E_{p+2}^{*b}, \ldots, E_N^{*b}\} \ \rightarrow\ Y_t^{*b} = \sum_{i=1}^{p} \hat{A}_i Y_{t-i}^{*b} + E_t^{*b}$$

FFT Bootstrap (FB)

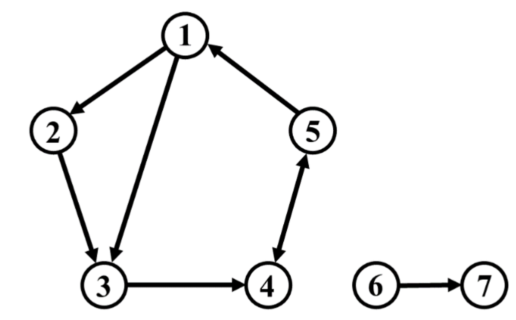
$$\mathcal{F}\{Y(t)\} = A(f)e^{i\phi(f)} \ \rightarrow\ Y_t^{*b} = \mathcal{F}^{-1}\left\{A(f)e^{i[\phi(f)+\varphi(f)]}\right\}$$

Random variable $[0, 2\pi]$

4 Simulation



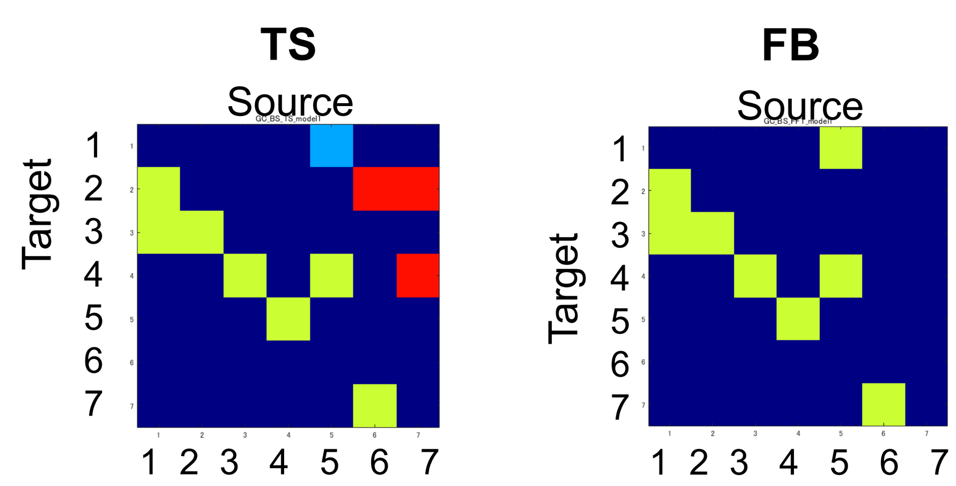
7 channels
1st~6th ch are oscillators
(0.1Hz, sampling rate 0.1sec [different damping rate for each channel])
7th ch is just gaussian noise
Modified the model proposed in [5]



Connectivity diagram

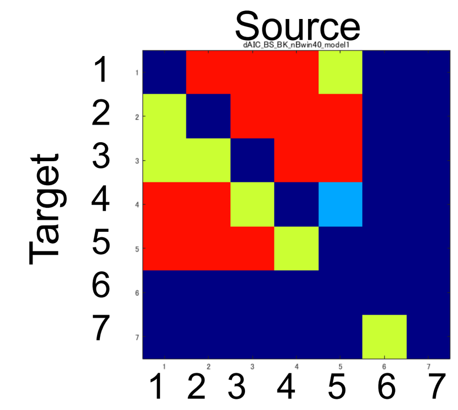
5 Results

Granger Causality (GC)



Difference of AIC (dAIC)

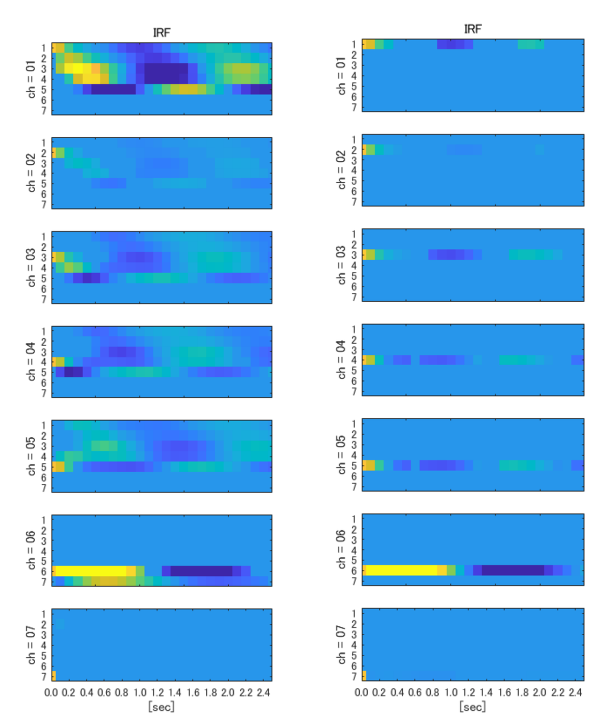
BK(block width = 40)



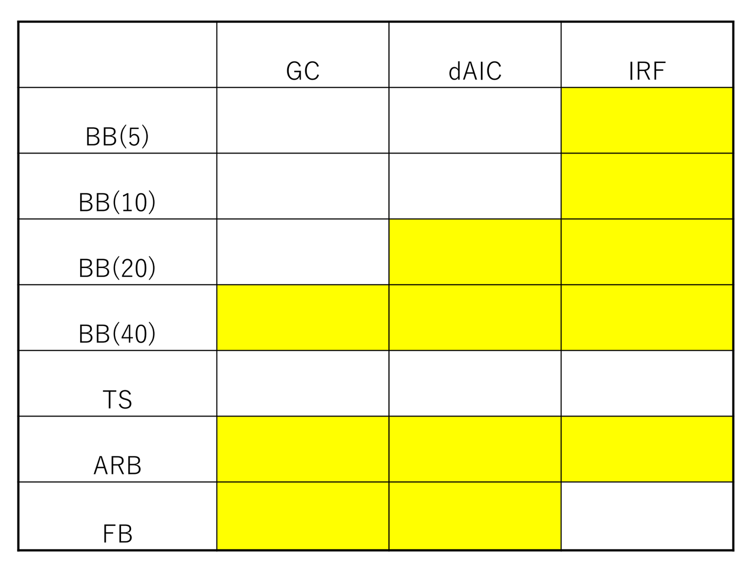
Blue; false negative
Red: false positive

Impulse Response (IR) Analysis

ARB　**TS**



Summary



BB(width of the block)
Yellow: correctly estimated

6 Conclusion

This study, using simulated data, practically demonstrated a combination of causality analysis and bootstrap methods that can properly estimate causal relations. The TS did not correctly detect causal relation using any causality analysis appropriately. GC and dAIC can be correctly estimated by BB, ARB, and FB.ARB and BB with all block widths correctly estimated IRF.

These results indicate BB(40) and ARB have the best estimation performance. However, because of the arbitrariness of the block width in BB, ARB is superior because it is less arbitrary (model order can be determined from real data using AIC, etc.).

Since GC cannot detect the temporal transition of the causal relation and IRF has difficulty obtaining spatial information, the best approach is to use both in a complementary manner.

References

[1] Granger, C. W. J. (1969) *Econometrica*. **37** (3): 424–438
[2] Kishino, H. & Hasegawa, M. (1989) *J. Mol. Evol.* 29, 170-179
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[4] Prichard, D. & Theller, J. (1994), *Phys. Rev. Lett.* 73(7) 951-954
[5] Sameshima, K., et al. (2015), *Brain. Inform.* 2, 119-133