# A bayesian method of second-order smoothing on Delaunay tessellation

牛源源(Niu Yuanyuan) 総合研究大学院大学 統計科学専攻 博士課程(3年次編入学) D3


## Abstract

For the non-uniqueness of solutions in geophysical inversion problems, prior information of the model is often required as the constraint condition. A common constraint information is to assume the smooth distribution of model parameters. Most of the geophysical data are irregularly distributed, however, most of the existing smoothing constraint methods are applied to the regular distribution of the observation data after gridding. This poster proposes a Bayesian smoothing constraint method based on Delaunay tessellation, which can guarantee the density of the original data and directly carry out the second-order smoothing constraint.


## Introduction

In this poster, firstly, Delaunay Tesselation is carried out for irregular gravity network. Secondly, based on the prior assumption of smooth distribution of Bouguer anomaly relative to surface relief, the quadratic polynomial function is used to fit the Bouguer gravity anomaly with gaussian noise in each triangle, and the smoothing constraints of the first and second derivatives are carried out at the same time. Finally, Bayesian optimal posterior estimation algorithm based on ABIC criterion is used to solve the model parameters iteratively.

## Methodology

**Objective function** Given some irregular distribution of gravity observation points, we do the delaunay tesselation on those position. As shown in Fig.1(a), the rectangular region is tessellated to $M$ triangles by the locations of $N$ observation points and some additional points on the boundaries including the corners. $\boldsymbol{B}$ denotes the $N$ observed Bouguer gravity anomaly values, which are obtained by sampling from the theoretical smooth Bouguer gravity anomaly data shown in Fig.1(b) and adding Gaussian noise.

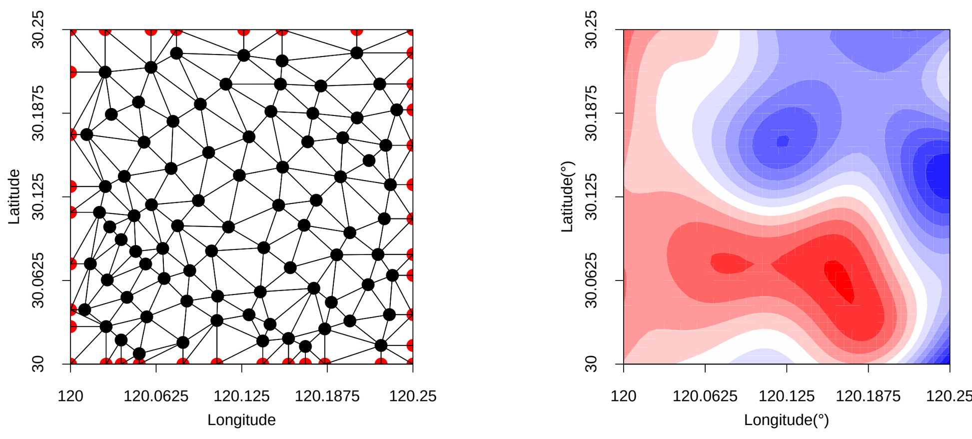


Figure 1: (a)Delaunay tesselation. (b)Theoretical smooth Bouguer gravity anomaly.

In order to smooth the observed Bouguer gravity anomaly, we choose the quadratic polynomial function to fit it in each delaunay triangle. Inside the $m$th triangle $\Delta_m(m = 1, ..., M)$, the smooth Bouguer gravity anomaly at any position $(x, y)$ is expressed by

$$B_m(x, y) = a_m\, x^2 + b_m\, x\, y + c_m\, y^2 + d_m\, x + e_m\, y + f_m. \tag{1}$$

Suppose that the smooth Bouguer gravity anomaly values at three vertices $B_{m1}, B_{m2}, B_{m3}$ and three mid-points $B_{m4}, B_{m5}, B_{m6}$ are known and take them into the Eq.(1), we can get

$$\tilde{\boldsymbol{B}}_m = \mathbf{E}_m\, \boldsymbol{\theta}_m, \tag{2}$$

$\tilde{\boldsymbol{B}}_m$ denotes the smooth Bouguer gravity anomaly of the three vertices and three mid-points in the $M$ tranigles.

With the assumption that the observed Bouguer gravity anomaly has Gaussian noise, the objective function can be written as

$$\boldsymbol{B} - \mathbf{C}\,\tilde{\boldsymbol{B}} = \boldsymbol{\varepsilon} \sim N\left(\mathbf{0}, \boldsymbol{\Sigma}_d^{-1}\right) \tag{3}$$

where $\tilde{\boldsymbol{B}}$ denotes $N$ smooth Bouguer gravity anomaly values, $\mathbf{C}$ denotes the configuration matrix, and

$$\boldsymbol{\Sigma}_d = \mathbf{W}_d^T\, \mathbf{W}_d = \begin{bmatrix} \frac{1}{\sigma_d^2} & & & \\ & \frac{1}{\sigma_d^2} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_d^2} \end{bmatrix}. \tag{4}$$

Then the conditional probability density function of the observed data is given by

$$L\left(\left[\boldsymbol{B} - \mathbf{C}\,\tilde{\boldsymbol{B}}\right] | \tilde{\boldsymbol{B}}\right) = \left[det\left(2\pi\left(\mathbf{W}_d^T\mathbf{W}_d\right)^{-1}\right)\right]^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\boldsymbol{B} - \mathbf{C}\,\tilde{\boldsymbol{B}}\right)^T \mathbf{W}_d^T\mathbf{W}_d \left(\boldsymbol{B} - \mathbf{C}\,\tilde{\boldsymbol{B}}\right)\right] \tag{5}$$

**Smooth constrain** With the prior information that Bouguer gravity anomaly is smooth distribute, we fit it by quadratic polynomial function and contrain its first-order flatness and second-order smoothness. $\mathbf{D}_1, \mathbf{D}_2$ denote the flatness matrix and smoothness matrix respectively. The prior knowledge can be expressed by the following normal distribution:

$$\begin{bmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{bmatrix} \tilde{\boldsymbol{B}} \sim N\left(\mathbf{0}, \begin{bmatrix} \boldsymbol{\Sigma}_1 & \\ & \boldsymbol{\Sigma}_2 \end{bmatrix}^{-1}\right) \tag{6}$$

The prior probability density function is given by:

$$\pi(\tilde{\boldsymbol{B}}) = (2\pi)^{-\frac{rank(\boldsymbol{D}^T\boldsymbol{D})}{2}} \left[det\left(\frac{\sigma_d^2}{\boldsymbol{D}^T\boldsymbol{D}}\right)_+\right]^{-\frac{1}{2}} \exp\left[-\frac{\tilde{\boldsymbol{B}}^T\left(w_1\, \mathbf{D}_1^T\mathbf{D}_1 + w_2\, \mathbf{D}_2^T\mathbf{D}_2\right)\tilde{\boldsymbol{B}}}{2\sigma_d^2}\right] \tag{7}$$

where

$$\mathbf{D} = \begin{bmatrix} \sqrt{w_1}\, \mathbf{D}_1 \\ \sqrt{w_2}\, \mathbf{D}_2 \end{bmatrix}. \tag{8}$$

$w_1$ is the constrain weight between fitness and flatness and $w_2$ is the constrain weight between fitness and smoothness. To define the $\mathbf{D}_1, \mathbf{D}_2$, we use the followning penalty function:

$$\begin{aligned} \Phi(x, y) = \iint_\Omega & w_1 \left\{\left[\frac{\partial B_m(x, y)}{\partial x}\right]^2 + \left[\frac{\partial B_m(x, y)}{\partial y}\right]^2\right\} + \\ & w_2 \left\{\left[\frac{\partial^2 B_m(x, y)}{\partial x^2}\right]^2 + 2\left[\frac{\partial^2 B_m(x, y)}{\partial xy}\right]^2 + \left[\frac{\partial^2 B_m(x, y)}{\partial y^2}\right]^2\right\} dxdy. \end{aligned} \tag{9}$$

**Bayesian optimal posterior estimation and ABIC** According to the Eq.5 and Eq.7 and Bayes formula, the posterior probability density function of model parameters can be expressed as

$$p\left(\tilde{\boldsymbol{B}}\,|\,\boldsymbol{B} - \mathbf{C}\,\tilde{\boldsymbol{B}}\right) = \frac{L\left(\boldsymbol{B} - \mathbf{C}\,\tilde{\boldsymbol{B}}\,|\tilde{\boldsymbol{B}}\right)\,\pi\left(\tilde{\boldsymbol{B}}\right)}{p\left(\boldsymbol{B} - \mathbf{C}\,\tilde{\boldsymbol{B}}\right)}. \tag{10}$$

We use the ABIC method to determine the value of hyper-parameters and goodness of the posterior model

$$\begin{aligned} ABIC = \min \Big[ & \left(N - rank\left(\mathbf{D}^T\mathbf{D}\right)\right) \log 2\pi\hat{\sigma}_d^2 - \log\left(\det\left[\mathbf{D}^T\mathbf{D}\right]_+\right) \\ & + \log\left(\det\left[\mathbf{Z}^T\mathbf{Z}\right]\right) + \left(N - rank\left(\mathbf{D}^T\mathbf{D}\right)\right)\Big] + 4 \end{aligned} \tag{11}$$

where

$$\hat{\sigma}_d^2 = \frac{\mathbf{U}\left(\tilde{\mathbf{B}}^*\right)}{N - rank\left(\mathbf{D}^T\mathbf{D}\right)}. \tag{12}$$

Here, $\tilde{\boldsymbol{B}}^*$ is the optimal solution of $\mathbf{U}\left(\tilde{\mathbf{B}}\right) = \|\boldsymbol{a} - \mathbf{Z}\tilde{\boldsymbol{B}}\|^2$, where $\boldsymbol{a} = \begin{bmatrix} \boldsymbol{B} \\ \mathbf{0} \end{bmatrix}$, $\mathbf{Z} = \begin{bmatrix} \mathbf{C} \\ \mathbf{D} \end{bmatrix}$, and

$$\tilde{\boldsymbol{B}}^* = \left(\mathbf{Z}^T\mathbf{Z}\right)^{-1} \mathbf{Z}^T\, \boldsymbol{a}. \tag{13}$$

## Synthetic data test

After adding different levels of Gaussian noise to the theoretical smooth gravity anomaly, the corresponding hyper-parameters are obtained by ABIC method, and then the estimated smooth Bouguer gravity anomaly data are calculated.

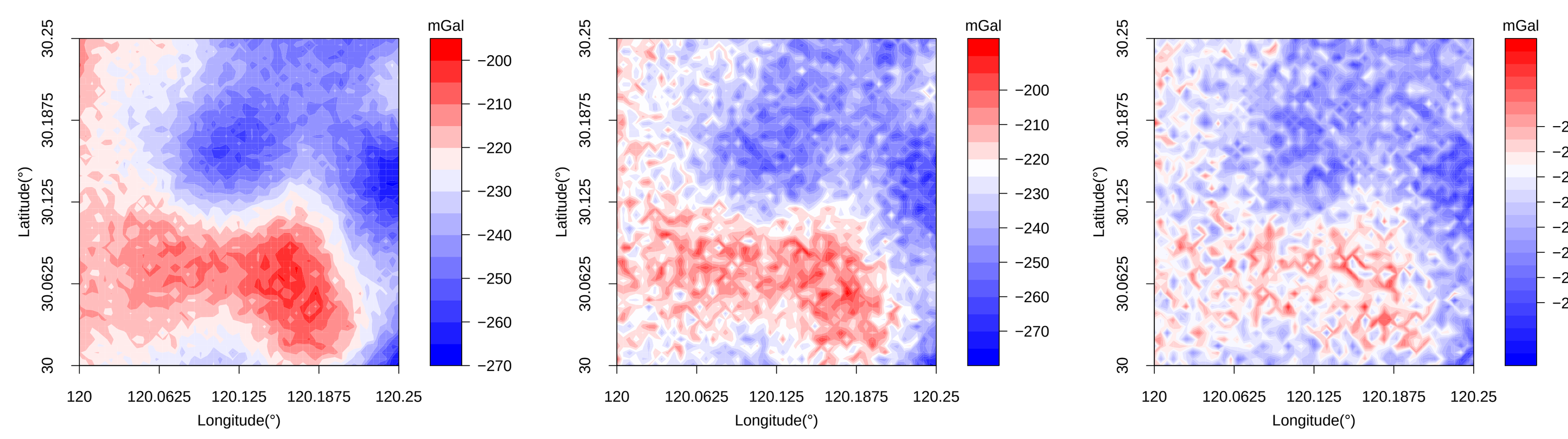


Figure 2: Observed data with (a)1% (b)3% (c)5% Gaussian noise.

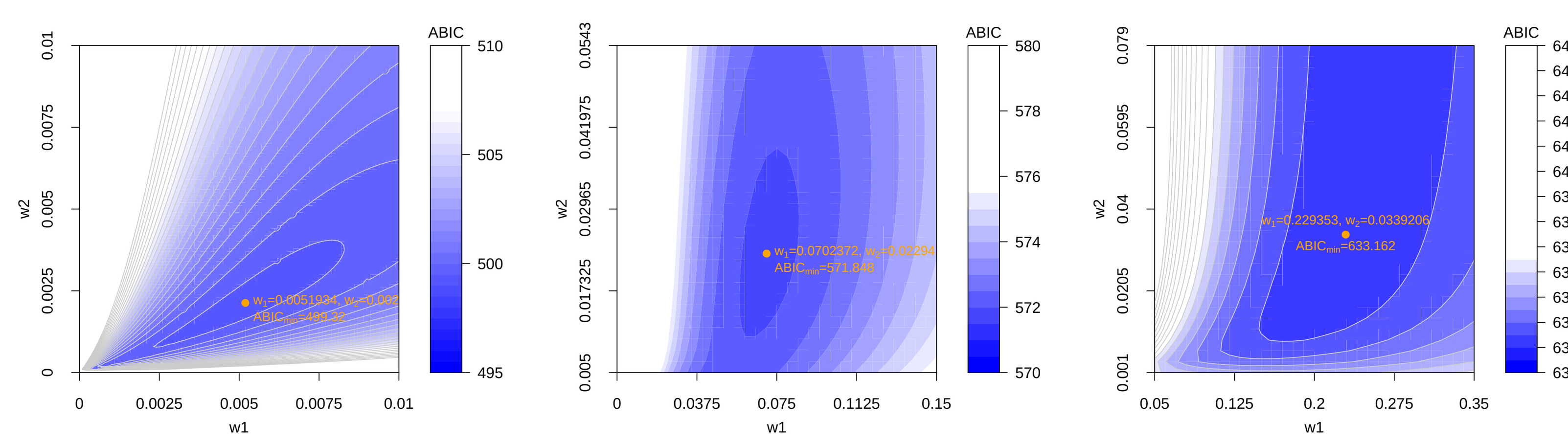


Figure 3: Hyper-parameters when the Gaussian noise level is (a)1% (b)3% (c)5%.

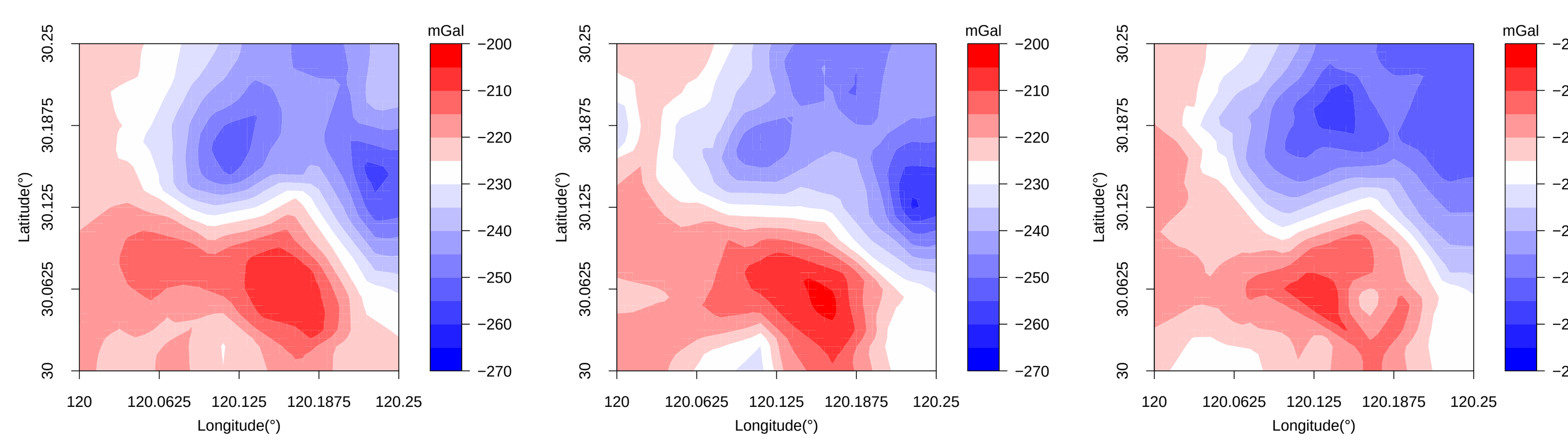


Figure 4: Estimated smooth Bouguer gravity anomaly when the Gaussian noise level is (a)1% (b)3% (c)5%.

## Conclusion and next work

The Bayesian method of second-order directly smoothing on Delaunay Tesselation can effectively smooth the noise. This method can be used in many geophysical inversion work, such as the gravity inversion and crustal strain-rate fields estimation and so on. The next step is to test the effect of simultaneous estimation of near-surface density and Bouguer gravity anomaly.