Geometry of amenable cones

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Goal of this project and further information

We prove several properties of amenable cones and compare amenability with other notions of facial exposedness. This is a joint work with Vera Roshchina (UNSW) and James Saunderson (Monash University) For more details, please check our arXiv preprints [2] and [3].

1 Convex sets and their faces

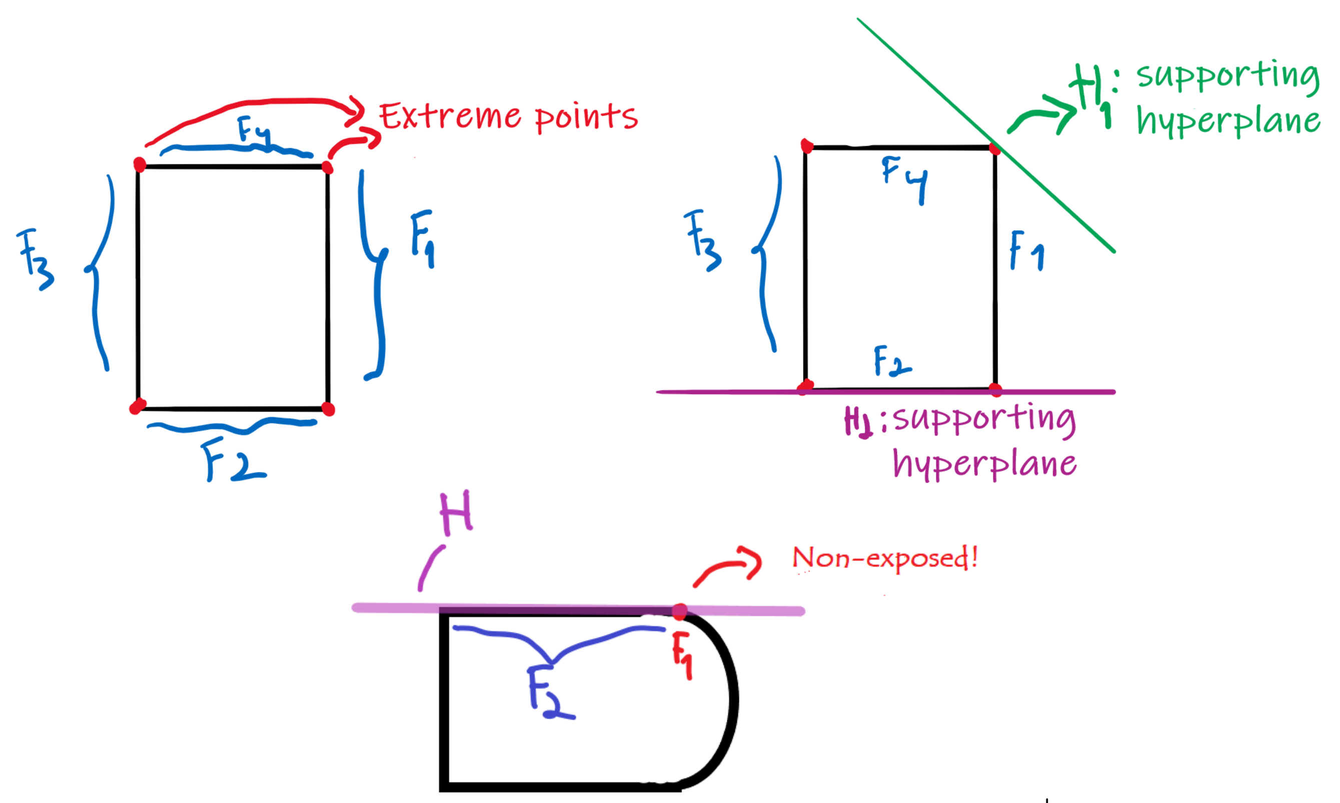
First, some definitions. Let

- C: a closed convex set contained in $\mathbb{R}^n$.
- F: closed convex set contained in C

F is a **face** of C (i.e., $F \trianglelefteq C$) $\overset{\text{def}}{\iff}$ if $\alpha x + (1-\alpha)y \in F$, with $x, y \in C$, $\alpha \in (0,1)$, then $x, y \in F$.

Extreme points are faces consisting of a single point.

A face F is **exposed** if there exists a supporting hyperplane H of C such that $F = C \cap H$. Examples.



Let $\mathcal{K}$ be a closed convex cone. We write $\mathcal{K}^*$, $\mathcal{K}^\perp$ and $\operatorname{span}\mathcal{K}$, for the dual cone, orthogonal complement and linear span of $\mathcal{K}$, respectively. We also define

$$\operatorname{dist}(x, \mathcal{K}) \coloneqq \inf\{\|x-y\| \mid y \in \mathcal{K}\}.$$

1.1 Notions of facial exposedness

Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex cone. We have the following definitions.

- $\mathcal{K}$ is **amenable** $\overset{\text{def}}{\iff}$ for every face $\mathcal{F} \trianglelefteq \mathcal{K}$ there is $\kappa > 0$ such that
$$\operatorname{dist}(x, \mathcal{F}) \leq \kappa \operatorname{dist}(x, \mathcal{K}), \qquad \forall x \in \operatorname{span}\mathcal{F}.$$
- $\mathcal{K}$ is **projectionally exposed** $\overset{\text{def}}{\iff}$ for every face $\mathcal{F} \trianglelefteq \mathcal{K}$ there exists a linear map $P : \mathbb{R}^n \to \mathbb{R}^n$ such that $P(\mathcal{K}) = \mathcal{F}$ and $P^2 = P$.
- $\mathcal{K}$ is **nice** $\overset{\text{def}}{\iff}$ $\forall \mathcal{F} \trianglelefteq \mathcal{K}$, $\mathcal{F}^* = \mathcal{K}^* + \mathcal{F}^\perp$.
- $\mathcal{K}$ is **facially exposed** $\overset{\text{def}}{\iff}$ every face is facially exposed.

These notions have many applications in the study of duality theory, facial reduction and representability in conic programming. Here we focus on **amenable cones**, which were proposed in [1] in order to study error bounds for conic systems.

2 Properties of amenable cones

Let $\mathcal{K}_1, \mathcal{K}_2$ be closed convex cones. The following results were proved inS

- If $\mathcal{K}_1$ and $\mathcal{K}_2$ are amenable, then $\mathcal{K}_1 \cap \mathcal{K}_2$ and $\mathcal{K}_1 \times \mathcal{K}_2$ are amenable.
- If A is an injective linear map and $\mathcal{K}_1$ is amenable then $A(\mathcal{K}_1)$ is amenable.
- Polyhedral cones and spectrahedral cones are amenable.
- Hyperbolicity cones are amenable.

In particular, second-order cones, positive semidefinite cones and all homogeneous cones are amenable.

3 Slicing amenable cones

Amenability can be defined for general convex sets as follows.

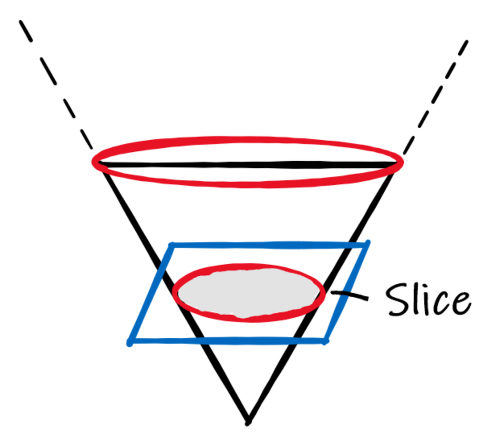
$C \subseteq \mathbb{R}^n$ is amenable $\overset{\text{def}}{\iff}$ for every $F \trianglelefteq C$ and every bounded set $B \subset \mathbb{R}^n$, there is $\kappa > 0$ such that the affine hull of F (denoted by $\operatorname{aff} F$) satisfy

$$\operatorname{dist}(x, F) \leq \kappa \operatorname{dist}(x, C), \qquad \forall x \in B \cap \operatorname{aff} F.$$

Let $\mathcal{K}$ be an amenable cone. A **slice of** $\mathcal{K}$ is an intersection of the form $\mathcal{K} \cap V$, where V is an affine space. The following statements hold.

- Every slice of $\mathcal{K}$ is an amenable convex set.
- If C is a amenable compact convex set, then the convex cone generated by $\{1\} \times C$ is amenable.

See [3, Section 4] for details.



A slice of a closed convex cone.

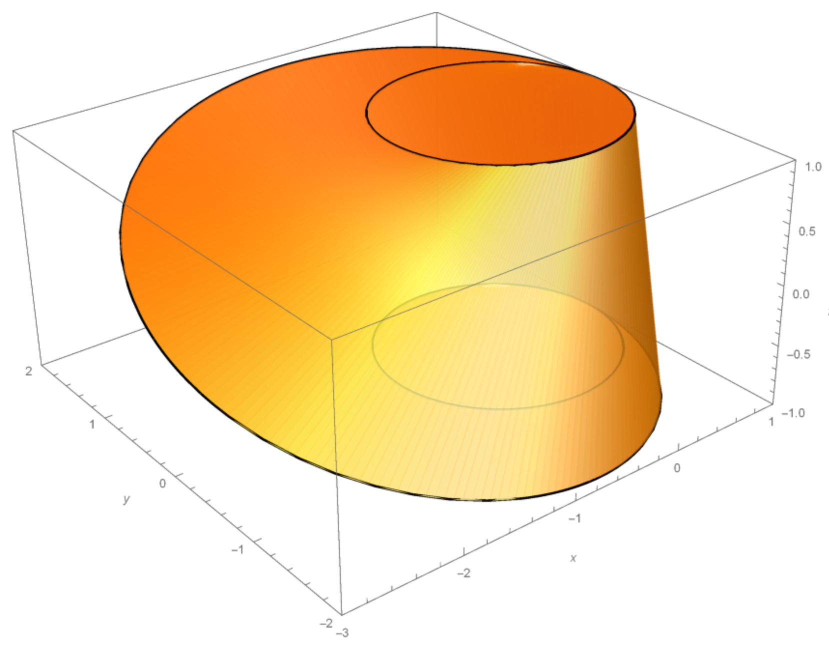
4 Comparison of exposedness properties

Known results:

- Facially exposed $\Leftarrow$ Nice $\Leftarrow$ Amenable $\Leftarrow$ Projectionally exposed.
- In dimension 3 or less: Facially exposed $\Leftrightarrow$ Projectionally exposed
- There exists a 4D cone that is facially exposed but not nice. [4]

New results (see Sections 5 and 6 in [2]):

- There exists a 4D cone that is nice but not amenable, see figure below.
- In dimension 4 or less: Amenable $\Leftrightarrow$ Projectionally exposed.



A 3D slice of a 4D convex cone that is nice but not amenable

References

[1] B. F. Lourenço. Amenable cones: error bounds without constraint qualifications. *Mathematical Programming*, 186:1–48, March 2021. arXiv:1712.06221.

[2] B. F. Lourenço, V. Roshchina, and J. Saunderson. Amenable cones are particularly nice. *arXiv e-prints*, November 2020. arXiv:2011.07745.

[3] B. F. Lourenço, V. Roshchina, and J. Saunderson. Hyperbolicity cones are amenable. February 2021. arXiv:2102.06359.

[4] V. Roshchina. Facially exposed cones are not always nice. *SIAM J. Optim.*, 24(1):257–268, 2014.