A bias-reduced GARCH-EVT (Extreme Value Theory) approach for financial risk estimation

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Main findings

Construction of a novel way to estimate one-step ahead **conditional Value-at-Risk** (**VaR**) based on the **Extreme Value Theory (EVT)**.
- Our approach (**GARCH-UGH**) is a two-step approach which consists of GARCH step to provide the volatility measures for heteroscedastic financial time series and UGH (EVT) step for the bias-reduced estimation of the tail of the distribution.
- **GARCH-UGH** outperformed other VaR estimation methods including standard **GARCH-EVT** in the empirical out-of-sample backtesting.

Background

The fluctuations of stock prices are relatively small and are often assumed to be normally distributed at longer time horizons. However, sometimes theses fluctuations can become catastrophic, especially when unforeseen large drops in prices are observed that could result in huge losses such as 2007-2008 crisis.

Two-fold Problem: **Stochastic volatility** and **heavy-tailedness** of the financial time series, which are known as the stylized facts.
- GARCH-type models assuming conditional normality are not well suited.
- Classical EVT models require i.i.d. assumption, which is violated.

Purpose: To evaluate potential losses occurring with extremely low probabilities, i.e., to estimate **Value-at-Risk (VaR)** which is just a quantile ($q_\tau = \inf\{x : F(x) \geq \tau\}$, $\forall \tau \in (0, 1)$) of the loss distribution with a small probability $p = 1 - \tau$. More precisely, **to introduce an alternative conditional VaR estimation** approach that copes with above listed problems.

Problem setting

Let $X_t = p_t - p_{t-1}$ be the log-returns (financial time series) where p_t is the logarithmic price at time t.
Assume the dynamics of X are

$$X_t = \mu_t + \sigma_t Z_t \quad \text{(Today)}$$

where μ_t, σ_t are mean and volatility and Z_t are i.i.d. innovations.
We are principally interested in 1-step ahead behaviour,

$$X_{t+1} = \mu_{t+1} + \sigma_{t+1} Z_{t+1} \quad \text{(Tomorrow)}.$$

Target: 1-step ahead **conditional VaR** at extreme levels

$$q_\tau(X_{t+1} \mid \mathscr{F}_t) = \mu_{t+1} + \sigma_{t+1} q_\tau(Z) \tag{1}$$

where we assume μ_{t+1} and σ_{t+1} are measurable w.r.t. the past $\mathscr{F}_t$, and Z has the same marginal distribution as of Z_t.

GARCH-UGH

McNeil and Frey (2000) first used a GARCH-type model for filtering and fit Generalized Pareto distribution (GPD) known as parametric Peaks-Over-threshold method to estimate an equation (1). The **GARCH-EVT** approach.

Our approach: We use the semiparametric **bias-reduced Hill estimator** instead of POT approach in EVT step. We call it **GARCH-UGH**.

GARCH step (Target is $\hat{\mu}_{t+1}$, $\hat{\sigma}_{t+1}$):

1. Fit AR(1)-GARCH(1,1) model to the return data by the quasi-maximum likelihood (QML) approach.
   - Conditional variance of the mean-adjusted series $\epsilon_t = X_t - \mu_t$ is

   $$\sigma_t^2 = \lambda_0 + \lambda_1 \epsilon_{t-1}^2 + \lambda_2 \sigma_{t-1}^2, \quad (\lambda_0, \lambda_1, \lambda_2 > 0). \quad \text{(GARCH(1,1)}$$

   - Conditional mean is given by

   $$\mu_t = \phi X_{t-1}. \quad \text{(AR(1))}$$

   - The likelihood for GARCH(1,1) model with normal innovations is maximized to obtain $\hat{\theta} = (\hat{\phi}, \hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2)$.
2. **Residuals** are calculated as

   $$(\hat{Z}_{t-n+1}, \ldots, \hat{Z}_t) = \left(\frac{x_{t-n+1} - \hat{\mu}_{t-n+1}}{\hat{\sigma}_{t-n+1}}, \ldots, \frac{x_t - \hat{\mu}_t}{\hat{\sigma}_t} \right).$$

   - regarded as i.i.d. and are used to estimate $q_\tau(Z)$ in equation (1).
3. Estimate 1-step ahead conditional mean and volatility by

   $$\hat{\mu}_{t+1} = \hat{\phi} x_t, \quad \hat{\sigma}_{t+1} = \sqrt{\hat{\lambda}_0 + \hat{\lambda}_1 \hat{\epsilon}_t^2 + \hat{\lambda}_2 \hat{\sigma}_t^2}.$$

GARCH-UGH Cont.

UGH step (Target is $\hat{q}_\tau(Z)$):
Use **top k order statistics** above the random threshold $\hat{Z}_{n-k,n}$, which need to be an upper intermediate (i.e. $k = k_n$ with $k \to \infty$ and $k/n \to 0$ as $n \to \infty$).

1. Let $\hat{Z}_{1,n} \leq \cdots \leq \hat{Z}_{n,n}$ denote the order values of $(\hat{Z}_{t-n+1}, \ldots, \hat{Z}_t)$.
2. Let m the number of positive observations in the sample n.
3. For each sample fraction k satisfying $k \leq \min(m - 1, \frac{2m}{\log\log m})$, and $\alpha = 1, \ldots, 4$, calculate the statistics and estimators given below;

$$M_k^{(\alpha)} = \frac{1}{k} \sum_{i=1}^{k} (\log Z_{n-i+1,n} - \log Z_{n-k,n})^\alpha,$$

$$S_k^{(2)} = \frac{3}{4} \frac{(M_k^{(4)} - 24(M_k^{(1)})^4)(M_k^{(2)} - 2(M_k^{(1)})^2)}{M_k^{(3)} - 6(M_k^{(1)})^3},$$

$$\hat{\rho}_k = \frac{-4 + 6S_k^{(2)} + \sqrt{3S_k^{(2)} - 2}}{4S_k^{(2)} - 3}, \quad \text{provided} \quad S_k^{(2)} \in \left(\frac{2}{3}, \frac{3}{4}\right),$$

$$\hat{\gamma}_{k,k_\rho} = \hat{\gamma}_k^H - \frac{M_k^{(2)} - 2(\hat{\gamma}_k^H)^2}{2\hat{\gamma}_k^H \hat{\rho}_{k_\rho} (1 - \hat{\rho}_{k_\rho})^{-1}} \quad \text{given} \quad M_k^{(1)} = \hat{\gamma}_k^H \quad \text{is Hill estimator},$$

$$\hat{q}_\tau(Z) = \hat{Z}_{n-k,n} \left(\frac{k}{np} \right)^{\hat{\gamma}_{k,k_\rho}} \left(1 - \frac{[M_k^{(2)} - 2(\hat{\gamma}_k^H)^2][1 - \hat{\rho}_{k_\rho}]^2}{2\hat{\gamma}_k^H \hat{\rho}_{k_\rho}^2} \left(1 - \left(\frac{k}{np} \right)^{\hat{\rho}_{k_\rho}} \right) \right),$$

where γ is the **extreme value index** (i.e. shape parameter), ρ is the second-order parameter and $\hat{\rho}_{k_\rho}$ is the one optimal $\hat{\rho}_k$ selected following the recommendation given in de Haan *et al.* (2016).

Real data analysis

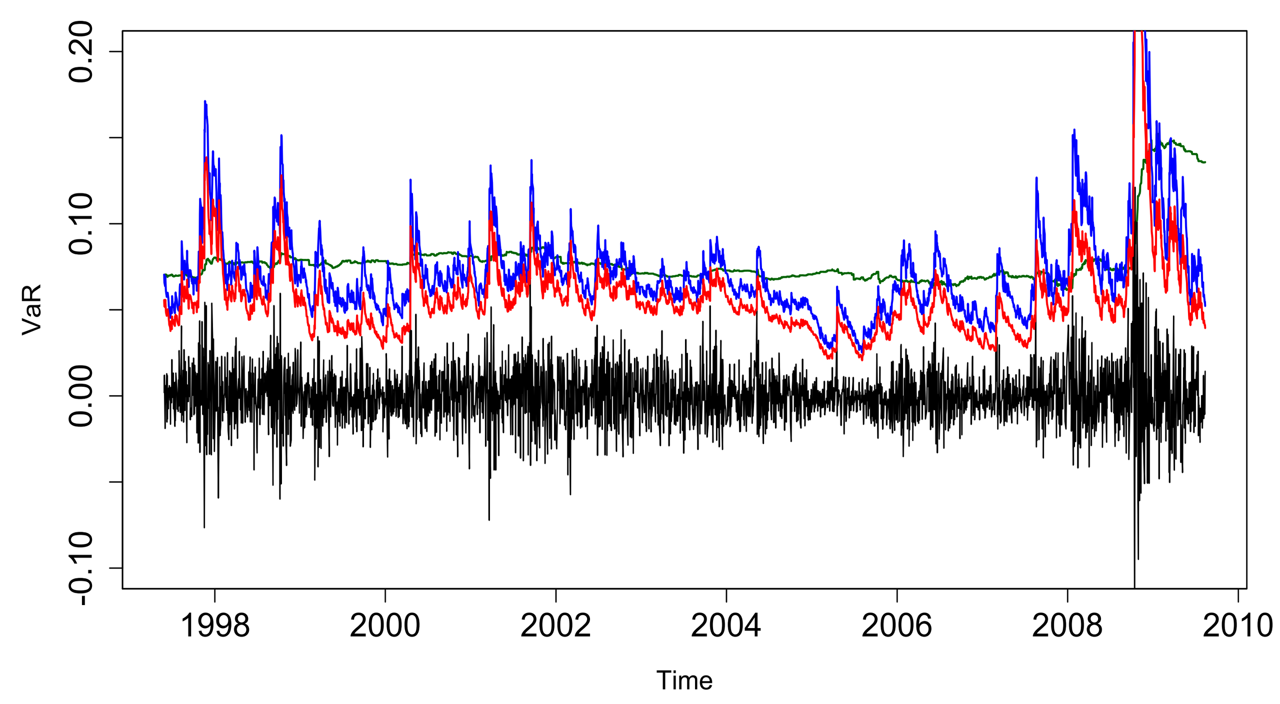

Figure 1: **Out-of-sample backtesting** of the **NIKKEI** index from 1997-05-29 to 2009-08-12, and showing the 99.9%-VaR estimates calculated using the historical period 1993-05-14 to 1997-05-28 and when top 20% of observations based from this period. The estimators are our GARCH-UGH, (blue line), standard GARCH-EVT, (red line) and UGH only (green line) superimposed on the negative log-returns (black line).

Testing window	3000				
Estimation window	1000				
% of top obs. used	5%	10%	15%	20%	25%
0.999 Quantile					
Expected	3	3	3	3	3
UGH	7	6	5	5	**3**
	(0.049, 0.142)	(0.128, 0.310)	(0.292, 0.569)	(0.292, 0.569)	(1.000, 0.997)
GARCH-UGH	**4**	**2**	**2**	**3**	1
	(0.583, 0.855)	(0.538, 0.826)	(0.538, 0.826)	(1.000, 0.997)	(0.179, 0.406)
GARCH-EVT	5	**4**	6	6	6
	(0.292, 0.569)	(0.583, 0.855)	(0.128, 0.310)	(0.128, 0.310)	(0.128, 0.310)

Table 1: Closer numbers of **VaR violations** (i.e. observations > VaR) to theoretically expected ones are highlighted in bold. The p-values for the unconditional coverage test by Kupiec (1995) and conditional coverage test by Christoffersen (1998) at 5% significance level are given in brackets in order.

McNeil, A.J. & Frey, R. (2000). Journal of Empirical Finance.
de Haan, L., Mercadier, C. & Zhou, C. (2016). Finance and Stochastics.