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Trace methods for hermitian K–theory

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1. Introduction.

Algebraic K-theory is an invariant of rings and schemes deeply related to arithmetic and algebraic geometry, but extremely difficult to compute. Still, calculations based on the approximation of algebraic K-theory by cyclic homology, or on the trace methods, proved fruitful in investigating the local nature of the invariant. In this note I explain my current work in progress to develop a similar methodology for calculating hermitian K-theory, a variant of algebraic K-theory defined for objects equipped with duality.

2. Background: Trace methods for algebraic K-theory

Denote by $\infty\mathrm{Cat}^{\mathrm{st}}$ the (large) ∞-category of stable ∞-categories and exact ∞-functors. The ∞-category NMot of *noncommutative motives* ([BGT]), together with the ∞-functor $z : \infty\mathrm{Cat}^{\mathrm{st}} \to \mathrm{NMot}$, is universal among ∞-functors $s : \infty\mathrm{Cat}^{\mathrm{st}} \to \mathcal{M}$ satisfying certain conditions, one of which requires that, roughly, if $C \to \mathcal{D} \to \mathcal{E}$ is a cofibre sequence of stable ∞-categories then $s(C) \to s(\mathcal{D}) \to s(\mathcal{E})$ is a cofibre sequence in $\mathcal{M}$. By definition, the (nonconnective) *algebraic K-theory* $\mathbf{K}(C)$ of an ∞-category C is the mapping spectrum $\mathrm{map}_{\mathrm{NMot}}(z(\mathrm{Perf}_{\mathbf{S}}), z(C))$, where $\mathbf{S}$ denotes the sphere spectrum and for a ring spectrum R, Perf_R denotes the stable ∞-category of compact objects in the stable ∞-category of R-module spectra. The homotopy groups $\mathbf{K_n}(C) = \pi_n(\mathbf{K}(C))$ are the *algebraic K-groups*.

[**Example.** Let $C = \mathrm{Perf}_R$, the stable ∞-category of complexes of modules over a discrete ring R, quasi-isomorphic to a bounded complex of finitely generated projective modules. Then the 0-th K-group $\mathbf{K_0}(\mathrm{Perf}_R)$ is the free abelian group generated by the set of quasi-isomorphism classes of such complexes modulo the relation that identifies the class $[E]$ with the sum of the classes $[E']$ and $[E'']$ if there is a distinguished triangle $E' \to E \to E'' \to E'[1]$.]

In general, the higher algebraic K-groups $\mathbf{K_n}(\mathrm{Perf}_R)$ are extremely difficult to compute. But it is known that K-theory can be approximated by cyclic homology: For a ring R, there is a canonical S^1-equivariant map $\mathbf{K}(R) := \mathbf{K}(\mathrm{Perf}_R) \to \mathrm{HH}(R)$, where the circle group S^1 acts trivially on the domain and the target denotes the Hochschild homology spectrum of R considered with the canonical action by S^1. The induced map $\mathrm{tr} : \mathbf{K}(R) \to \mathrm{HC}^-(R)$ is called the *trace map*, where $\mathrm{HC}^-(R)$ denotes the homotopy fixed point spectrum of $\mathrm{HH}(R)$ under the S^1-action, called the *negative cyclic homology spectrum*.

[**Theorem** (Goodwillie [G]). Let R be a discrete ring and I a nilpotent ideal thereof. Write $\mathbf{K}(R, I)_{\mathbf{Q}}$ and $\mathrm{HC}^-(R_{\mathbf{Q}}, I)$ for the homotopy fibres of the maps $\mathbf{K}(R)_{\mathbf{Q}} \to \mathbf{K}(R/I)_{\mathbf{Q}}$ and $\mathrm{HC}^-(R_{\mathbf{Q}}) \to \mathrm{HC}^-((R/I)_{\mathbf{Q}})$, respectively, where $(\text{-})_{\mathbf{Q}}$ denotes rationalization. Then the trace map induces a homotopy equivalence from $\mathbf{K}(R, I)_{\mathbf{Q}}$ to $\mathrm{HC}^-(R_{\mathbf{Q}}, I)$.]

To remove the rationalization from the statement, one needs to replace the cyclic homological invariants by more sophisticated variants, i.e. $\mathrm{HH}(R)$ by the *topological Hochschild homology spectrum* $\mathrm{THH}(R)$ and $\mathrm{HC}^-(R)$ by the *topological cyclic homology spectrum* $\mathrm{TC}(R)$. The topological Hochschild homology spectrum admits a canonical action by S^1, and the topological cyclic homology spectrum is a certain refinement of its homotopy fixed point spectrum. There is a canonical S^1-equivariant map $\mathbf{K}(R) \to \mathrm{THH}(R)$, and the induced map $\mathrm{tr} : \mathbf{K}(R) \to \mathrm{TC}(R)$ is called the *cyclotomic trace map*.

[**Theorem** (Dundas-Goodwillie-McCarthy [DGM]). For a ring R and a nilpotent ideal I thereof, the cyclotomic trace map induces a homotopy equivalence from $\mathbf{K}(R, I)$ to $\mathrm{TC}(R, I)$, where $F(R, I)$ denotes the homotopy fibre of the map $F(R) \to F(R/I)$ for $F = \mathbf{K}$ or TC.]

3. Trace methods for hermitian K-theory.

Let C be a stable ∞-category equipped with a duality $(\text{-})^\vee$, i.e. an equivalence between C^{op} and C. I define the *nonconnective hermitian K-theory* $K^h(C; (\text{-})^\vee)$ to be the homotopy fixed point spectrum of the nonconnective algebraic K-theory $\mathbf{K}(C)$ with respect to the $\mathbf{Z}/2$-action induced by $(\text{-})^\vee$.

[**Conjecture**. The nonconnective hermitian K-theory spectrum in my sense agrees with Schlichting's Karoubi-Grothendieck-Witt spectrum [S].]

[**Example**. Let $C = \mathrm{Perf}_R$ and $(\text{-})^\vee$ be the contravariant autoequivalence thereon that associates the Hom-complex $\underline{\mathrm{Hom}}_R(E, R[0])$ to a perfect complex E, where $R[0]$ denotes the complex whose 0-th term is R and whose other terms are trivial. Then the 0-th hermitian K-group $K^h{}_0(R) = \pi_0 K^h(\mathrm{Perf}_R; (\text{-})^\vee)$ is the free abelian group generated by the set of perfect complexes E equipped with a quasi-isomorphism with its dual $\underline{\mathrm{Hom}}_R(E, R[0])$, modulo the relation that identifies the second term of a distinguished triangle $E' \to E \to E'' \to E'[1]$ with the sum of the first and third terms if the sequence is compatible with the duality quasi-isomorphisms on the respective terms.]

The aim of my work in progress is to establish the trace methods for calculating hermitian K-theory. To this end, first recall the following recent result by Nikolaus and Scholze.

[**Theorem** (Nikolaus-Scholze [NS]). The topological cyclic homology spectrum $\mathrm{TC}(C)$ of a stable ∞-category C is given by the mapping spectrum $\mathrm{map}_{\mathrm{CycSp}}(\mathrm{THH}(\mathrm{Perf}_{\mathbf{S}}), \mathrm{THH}(C))$, where CycSp is the stable ∞-category of *cyclotomic spectra*, which are by definition S^1-equivariant spectra equipped with a certain extra data called the *cyclotomic structures*.]

Note that if X is an S^1-equivariant spectrum then the automorphism of the *commutative* compact Lie group S^1 that maps z to z^{-1} gives rise to another S^1-action on X. This operation defines an autoequivalence $(\text{-})^{\mathrm{op}}$ on the stable ∞-category of S^1-equivariant spectra and similarly on CycSp. Write D-CycSp for the stable ∞-category of cyclotomic spectra X equipped with an equivalence with X^{op}.

[**Definition**. Let C be a stable ∞-category with duality $(\text{-})^\vee$, i.e. an equivalence with C^{op}. Then the mapping spectrum $\mathrm{map}_{\mathrm{D\text{-}CycSp}}(\mathrm{THH}(\mathrm{Perf}_{\mathbf{S}}), \mathrm{THH}(C))$, where $\mathrm{THH}(\mathrm{Perf}_{\mathbf{S}})$ is equipped with an equivalence with $\mathrm{THH}(\mathrm{Perf}_{\mathbf{S}})^{\mathrm{op}}$ induced by the duality that takes a finite spectrum X to the mapping spectrum $\mathrm{map}(X, \mathbf{S})$ and $\mathrm{THH}(C)$ equipped with the equivalence with $\mathrm{THH}(C)^{\mathrm{op}}$ induced by $(\text{-})^\vee$, is denoted by $\mathrm{TD}(C; (\text{-})^\vee)$ and called the *topological dihedral homology*.]

Note that the ∞-functor $\mathrm{THH} : \infty\mathrm{Cat}^{\mathrm{st}} \to \mathrm{CycSp}$ is compatible with the autoequivalences on the domain and target, both denoted by $(\text{-})^{\mathrm{op}}$.

[**Proposition**. For a stable ∞-category C with duality $(\text{-})^\vee$, write D-C for the stable ∞-category of objects E of C equipped with an equivalence φ with $E^\vee$. Then the mapping spectrum $\mathrm{map}_{\mathrm{D}\text{-}C}((E, \varphi), (E', \varphi'))$ is equivalent to the homotopy fixed point spectrum of $\mathrm{map}_C(E, E')$ with respect to the $\mathbf{Z}/2$-action given by the conjugation by φ and φ'. In particular, the cyclotomic trace map $\mathrm{tr} : \mathbf{K}(C) \to \mathrm{TC}(C)$ induces the *dihedro-cyclotomic trace map* $\mathrm{tr}^h : K^h(C; (\text{-})^\vee) \to \mathrm{TD}(C; (\text{-})^\vee)$.]

I am currently working to show the following hermitian analogue of the Dundas-Goodwillie-McCarthy theorem:

[**Goal**. Let R be a ring with anti-involution $w : R \to R^{\mathrm{op}}$ and I a nilpotent ideal preserved by w. Then the dihedro-cyclotomic trace map should induce an equivalence from $K^h(R, I; w)$ to $\mathrm{TD}(R, I; w)$.]

References.

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