Error bounds and convex feasibility problems

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Goal of this project and further information

The goal is to provide a general framework for error bounds and analyze the convergence of several algorithms for convex feasibility problems. This is a joint work with Tianxiang Liu (RIKEN-AIP). For more details, please check our arXiv preprint [3].

1 What is a convex feasibility problem?

$C_1, \ldots, C_m \subseteq \mathbb{R}^n$: closed convex sets.

$$\text{find } x \in C := \bigcap_{i=1}^{m} C_i, \qquad \text{(CFP)}$$

Some applications include: finding a feasible solution of an LP/SOCP/SDP; refining solutions of convex optimization problems obtained by iterative solvers; and several problems in image processing [1, 2].

2 Algorithms for CFPs

P_{C_i}: orthogonal projection onto C_i.
$\operatorname{dist}(x, C_i)$: Euclidean distance from x to C_i.

Suppose $C \neq \emptyset$ and let x^0 be an initial point.

(*a*) *Mean projection algorithm* (**MPA**): $x^{k+1} = \sum_{i=1}^m \frac{1}{m} P_{C_i}(x^k)$.

(*b*) *Cyclic projection algorithm* (**CPA**): $x^{k+1} = P_{C_{t(k)}}(x^k)$, where $t(k) := (k \bmod m) + 1$.

(*c*) *Adaptive weighted projection algorithm* (**AWPA**):

$$x^{k+1} = \sum_{i=1}^{m} \frac{d_i^k}{d_1^k + \cdots + d_m^k} P_{C_i}(x^k),$$

where $d_i^k := \operatorname{dist}(x^k, C_i)$.

(*d*) Damped versions of the Douglas-Rachford algorithm (**dampDR**).

See also [1] and [3, Section 4.3].

- **Do these methods always work? Yes, because of convexity**
- **How fast do they converge? Depends on the kind of error bound that holds between sets**

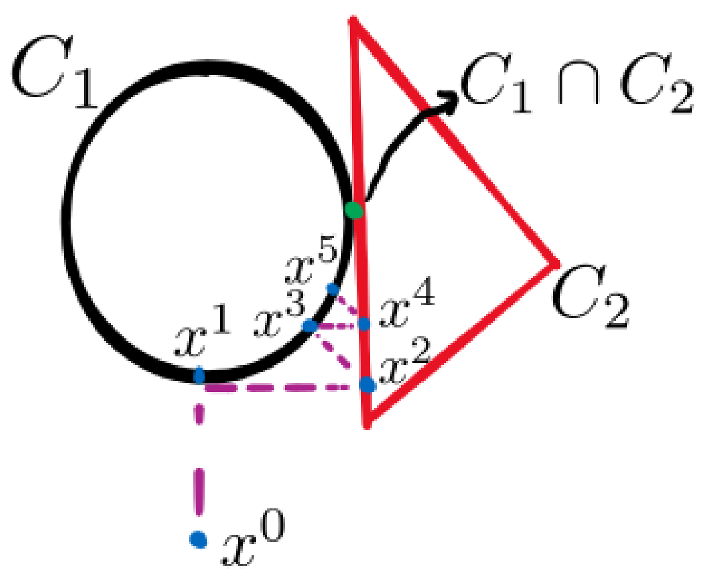

Figure1: Example of **CPA**.

3 A new framework for error bounds

Definition. $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a **consistent error bound function (CEBF)** for $C_1, \ldots, C_m$ if:

(*i*) the following error bound holds:

$$\operatorname{dist}(x, C) \leq \Phi\left(\max_{1 \leq i \leq m} \operatorname{dist}(x, C_i), \|x\|\right) \qquad \forall\, x \in \mathbb{R}^n;$$

(*ii*) $\forall b \geq 0$, $\Phi(\cdot, b)$ is monotone nondecreasing, right-continuous at 0 and $\Phi(0, b) = 0$;

(*iii*) $\forall a \geq 0$, $\Phi(a, \cdot)$ is monotone nondecreasing.

Some results:

- $C \neq \emptyset \Rightarrow \exists \Phi$ a **CEBF** for $C_1, \ldots, C_m$
- Lipschitzian, Hölderian error bounds and error bounds for **amenable cones** (see [4]) can be expressed as **CEBFs**.
- $C_1, \ldots, C_m$ satisfy an uniform Hölderian error bound with exponent γ $\Leftrightarrow$ $\exists \Phi$ a **CEBF** for $C_1, \ldots, C_m$ such that $\Phi(a, b) := \rho(b) a^\gamma$, where ρ is monotone nondecreasing.
- $C_1, \ldots, C_m$ satisfy a Lipschitzian error bound $\Leftrightarrow$ $\exists \Phi$ a **CEBF** for $C_1, \ldots, C_m$ such that $\Phi(a, b) := \rho(b) a$, where ρ is monotone nondecreasing.

4 Main convergence results

Let Φ be a strict **CEBF** (i.e., $\Phi(\cdot, b)$ is monotone increasing). For $\kappa > 0$ and $\delta > 0$ define

$$\phi_{\kappa,\Phi}(t) := \left(\Phi(\sqrt{t}, \kappa)\right)^2, \quad t \geq 0.$$

$$\Phi_\kappa^\spadesuit(t) := \int_\delta^t \frac{1}{\phi_{\kappa,\Phi}^-(s)} ds, \quad t \in \left(0, \sup \phi_{\kappa,\Phi}\right),$$

where f^- denotes a generalized inverse of f.

A few results:

1. The convergence rate of **MPA**, **CPA**, **AWPA**, **dampDR** and several other methods can be expressed in terms of $\Phi_\kappa^\spadesuit$ and its inverse. E.g., rate for **CPA**:

$$\operatorname{dist}(x^k, C) \leq \sqrt{(\Phi_{\hat{\kappa}}^\spadesuit)^{-1}\left(\Phi_{\hat{\kappa}}^\spadesuit(\operatorname{dist}^2(x^0, C)) - (k - m - (k \bmod m))/m^2\right)} \;\forall\, k \geq 2m.$$

2. Suppose $C_1, \ldots, C_m$ satisfy an uniform Hölderian error bound with exponent γ and let $\{x^k\}$ be generated by **MPA, CPA**, **AWPA** or **dampDR**. Then, there exists $M > 0$ such that

$$\operatorname{dist}(x^k, C) \leq \begin{cases} M\, k^{-\frac{1}{2(\gamma^{-1}-1)}} & \text{if } \gamma \in (0,1), \textbf{ (Sublinear convergence)} \\ M\, \theta^k & \text{if } \gamma = 1. \quad \textbf{(Linear convergence)} \end{cases}$$

3. When applied to conic feasibility problems over amenable cones (see [4]), convergence rates taking into the account the "level of regularity" of the problem can be obtained. For example, let $\mathcal{K}$ be a symmetric cone and $\mathcal{V}$ be an affine space with $\mathcal{K} \cap \mathcal{V} \neq \emptyset$. Let $\{x^k\}$ be generated by **MPA, CPA**, **AWPA** or **dampDR**, then there exists $M > 0$ such that

$$\operatorname{dist}(x^k, \mathcal{K} \cap \mathcal{V}) \leq \begin{cases} M\, k^{-\frac{1}{2\left(2^{d_S(\mathcal{K},\mathcal{V})}-1\right)}} & \text{if a constraint qualification is not satisfied,} \\ M\, \theta^k & \text{otherwise,} \end{cases}$$

where $d_S(\mathcal{K}, \mathcal{V})$ is the **singularity degree** of the problem.

参考文献

[1] H. H. Bauschke and J. M. Borwein. On projection algorithms for solving convex feasibility problems. *SIAM Review*, 38(3):367–426, 1996.

[2] P. Combettes. The convex feasibility problem in image recovery. volume 95 of *Advances in Imaging and Electron Physics*, pages 155 – 270. Elsevier, 1996.

[3] T. Liu and B. F. Lourenço. Convergence analysis under consistent error bounds. *ArXiv e-prints*, 2020. arXiv:2008.12968.

[4] B. F. Lourenço. Amenable cones: error bounds without constraint qualifications. *ArXiv e-prints. To appear in* Mathematical Programming, December 2017. arXiv:1712.06221.