逆ガウス型分布の母中央値に関して

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ABSTRACT

Some distributions for positive random variables have population median. But the median of the inverse Gaussian distribution cannot be expressed by explicitly. We evaluate it by upper and lower limits.

1 INTRODUCTION

Let X be an inverse Gaussian random variable $IG(\mu, c^2)$ with the probability element $f(x)dx$

$$f(x)dx = \frac{1}{\sqrt{2\pi}c}\left(\frac{x}{\mu}\right)^{-\frac{3}{2}} \exp\left[-\frac{1}{2c^2}\left(\sqrt{\frac{x}{\mu}} - \sqrt{\frac{\mu}{x}}\right)^2\right]\frac{dx}{\mu},$$

where $0 < x < \infty,\ 0 < \mu < \infty$ and $0 < c < \infty$.

The parameters μ and c are the population arithmetic mean μ_A and the coefficient of variation of X, respectively.

We have the cumulative distribution function of $IG(\mu, c^2)$ as follows;

$$F(x) = \Phi\left(\frac{1}{c}\left(\sqrt{\frac{x}{\mu}} - \sqrt{\frac{\mu}{x}}\right)\right) + e^{2/c^2}\Phi\left(-\frac{1}{c}\left(\sqrt{\frac{\mu}{x}} + \sqrt{\frac{x}{\mu}}\right)\right)$$

, where $\Phi(u)$ is the cumulative distribution function of the standard normal distribution.

Then, the population meadian μ_{median} for $IG(\mu, c^2)$ is satisfied with

$$F(\mu_{median}) = \frac{1}{2}$$

Therefore, the median for $IG(\mu, c^2)$ can not be determined analytically.

Let X be a lognormal random variable $LN(\mu, c^2)$ with the probability element, $f(x)dx$, expressed as follows:

$$\frac{1}{\sqrt{2\pi}c}\left(\frac{x}{\mu}\right)^{-1} \exp\left[-\frac{1}{2c^2}\left(\ln\left(\frac{x}{\mu}\right)\right)^2\right]\frac{dx}{\mu},$$

where $0 < x < \infty,\ 0 < \mu < \infty$, and $0 < c < \infty$.

Here, μ is also a median, and c is a population standard deviation of $\ln(X/\mu)$.

2 POPULATION MEADIAN FOR IG

We have the cumulative distribution function of $IG(\mu, c^2)$ as follows;

$$F(x) = \Phi\left(\frac{1}{c}\left(\sqrt{\frac{x}{\mu}} - \sqrt{\frac{\mu}{x}}\right)\right) + e^{2/c^2}\Phi\left(-\frac{1}{c}\left(\sqrt{\frac{\mu}{x}} + \sqrt{\frac{x}{\mu}}\right)\right),$$

where $\Phi(u)$ is the cumulative distribution function of the standard normal distribution.

Then, the population meadian μ_{median} for $IG(\mu, c^2)$ is satisfied with

$$F(\mu_{median}) = \frac{1}{2}.$$

$$\Phi(0) = \frac{1}{2}.$$

$$\Phi(0) - \Phi\left(\frac{1}{c}\left(\sqrt{\frac{\mu_{median}}{\mu_A}} - \sqrt{\frac{\mu_A}{\mu_{median}}}\right)\right) = e^{2/c^2}\Phi\left(-\frac{1}{c}\left(\sqrt{\frac{\mu_A}{\mu_{median}}} + \sqrt{\frac{\mu_{median}}{\mu_A}}\right)\right).$$

The right hand side eqution is always positive. Then we have the following formula.

$$\Phi(0) > \Phi\left(\frac{1}{c}\left(\sqrt{\frac{\mu_{median}}{\mu_A}} - \sqrt{\frac{\mu_A}{\mu_{median}}}\right)\right)$$

$\Phi(u)$ is the monotonically increasing function.

$$\sqrt{\frac{\mu_{median}}{\mu_A}} < \sqrt{\frac{\mu_A}{\mu_{median}}}.$$

Theorem $X \sim IG(\mu, c^2)$, then

$$\mu_{median} < \mu_A.$$

$$\frac{1}{2} - e^{\frac{2}{c^2}}\Phi\left(-\frac{2}{c}\right) \le \Phi\left(\frac{1}{c}\left(\sqrt{\frac{\mu_{median}}{\mu_A}} - \sqrt{\frac{\mu_A}{\mu_{median}}}\right)\right) < \frac{1}{2}.$$

$\Phi(u)$ is the monotonically increasing function, then $\Phi^{-1}(u)$ means the inverse function of $\Phi(u)$.

$$\Phi^{-1}\left(\frac{1}{2} - e^{\frac{2}{c^2}}\Phi\left(-\frac{2}{c}\right)\right) \le \frac{1}{c}\left(\sqrt{\frac{\mu_{median}}{\mu_A}} - \sqrt{\frac{\mu_A}{\mu_{median}}}\right) < 0.$$

$$k = \Phi^{-1}\left(\frac{1}{2} - e^{\frac{2}{c^2}}\Phi\left(-\frac{2}{c}\right)\right) < 0.$$

$$0 > \sqrt{\frac{\mu_{median}}{\mu_A}} - \sqrt{\frac{\mu_A}{\mu_{median}}} \ge kc.$$

$$\mu_{median}^2 - ((-k)^2c^2 + 2)\mu_A \cdot \mu_{median} + \mu_A^2 \le 0.$$

Theorem $X \sim IG(\mu, c^2)$, then its population medain μ_{median} is satisfied with the following formula.

$$\frac{1}{2}\left(k^2c^2 + 2 + kc\sqrt{k^2c^2 + 4}\right)\mu_A \le \mu_{median} < \mu_A.$$

$$k = \Phi^{-1}\left(\frac{1}{2} - e^{\frac{2}{c^2}}\Phi\left(-\frac{2}{c}\right)\right) < 0.$$

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