球面上のベクトル場の推定とその応用

中野 慎也　モデリング研究系 准教授

Localized basis functions for spherical vector field

- A divergence-free vector field can be represented by a stream function Ψ^{df} as follows:

$$\boldsymbol{V}^{df}(\boldsymbol{r}) = -\boldsymbol{e}_r \times \nabla \Psi^{df}.$$

- A curl-free vector field can be represented by a potential function Ψ^{cf} as follows:

$$\boldsymbol{V}^{cf}(\boldsymbol{r}) = -\nabla \Psi^{cf}.$$

- According to the Helmholtz theorem, an arbitrary vector field on a sphere can be written as a sum of a divergence-free field and a curl-free field. We can thus represent any vector field in the following form:

$$\boldsymbol{V}(\boldsymbol{r}) = \boldsymbol{V}^{df}(\boldsymbol{r}) + \boldsymbol{V}^{cf}(\boldsymbol{r}) = -\boldsymbol{e}_r \times \nabla \Psi^{df} - \nabla \Psi^{cf}.$$

- We expand the stream function Ψ^{df} and the potential function Ψ^{cf} by using localized basis functions ψ^{df} and ψ^{cf}:

$$\Psi^{df}(\boldsymbol{r}) = \sum_i w_i^{df} \psi^{df}(\boldsymbol{r}, \boldsymbol{r}_i), \quad \Psi^{cf}(\boldsymbol{r}) = \sum_i w_i^{cf} \psi^{cf}(\boldsymbol{r}, \boldsymbol{r}_i).$$

- Defining the following vector-valued basis functions:

$$\boldsymbol{v}^{df}(\boldsymbol{r}, \boldsymbol{r}_i) = -\boldsymbol{e}_r \times \nabla \psi^{df}(\boldsymbol{r}, \boldsymbol{r}_i), \quad \boldsymbol{v}^{cf}(\boldsymbol{r}, \boldsymbol{r}_i) = -\nabla \psi^{cf}(\boldsymbol{r}, \boldsymbol{r}_i),$$

  an arbitrary spherical field can be represented using these basis functions

$$\boldsymbol{V}(\boldsymbol{r}) = \sum_i w_i^{df}\left(-\boldsymbol{e}_r \times \nabla \psi^{df}(\boldsymbol{r}, \boldsymbol{r}_i)\right) + \sum_i w_i^{cf}\left(-\nabla \psi^{cf}(\boldsymbol{r}, \boldsymbol{r}_i)\right)$$
$$= \sum_i w_i^{df} \boldsymbol{v}^{df}(\boldsymbol{r}, \boldsymbol{r}_i) + \sum_i w_i^{cf} \boldsymbol{v}^{cf}(\boldsymbol{r}, \boldsymbol{r}_i).$$

Modeling of divergence-free vector field

- We employ spherical Gaussian functions for obtaining a set of basis functions

$$\psi^{df}(\boldsymbol{r}, \boldsymbol{r}_i) = \psi^{cf}(\boldsymbol{r}, \boldsymbol{r}_i) = \exp\left[\eta\left(\frac{\boldsymbol{r}\cdot\boldsymbol{r}_i}{R_i^2} - 1\right)\right] = \exp\left[\eta\left(\cos\theta' - 1\right)\right].$$

- We then obtain

$$\boldsymbol{v}^{df}(\boldsymbol{r}, \boldsymbol{r}_i) = (\eta \boldsymbol{r}_i \times \boldsymbol{r}) \exp\left[\eta\left(\frac{\boldsymbol{r}\cdot\boldsymbol{r}_i}{R^2} - 1\right)\right], \quad \boldsymbol{v}^{cf}(\boldsymbol{r}, \boldsymbol{r}_i) = \boldsymbol{e}_{\theta,i}(\eta \sin\Delta\theta) \exp\left[\eta\left(\frac{\boldsymbol{r}\cdot\boldsymbol{r}_i}{R^2} - 1\right)\right].$$

- Now we consider a divergence-free vector field (no source, no sink) and expand the field by using the divergence-free basis functions:

$$\boldsymbol{V}(\boldsymbol{r}) = \sum_i w_i^{df} \boldsymbol{v}^{df}(\boldsymbol{r}, \boldsymbol{r}_i).$$

- When we use the expansion of our basis functions

$$\boldsymbol{V}(\boldsymbol{r}) = \sum_i w_i \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{r}_i),$$

  the node points $\boldsymbol{r}_i$ can be placed arbitrarily.

- We placed 2500 node points randomly and uniformly distributed in the region above 40 degree in latitude, which approximates the following Monte Carlo convolution:

$$\boldsymbol{V}(\boldsymbol{r}) = \int_S w(\boldsymbol{r}) \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{r}') d\boldsymbol{r} = \int_0^{\pi/2}\int_0^{2\pi} w(\boldsymbol{r}) \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{r}') R^2 \sin\theta\, d\theta\, d\phi.$$

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Application to ionospheric physics

- Now we apply this method for estimating the ionospheric plasma velocity distribution which can be assumed to be divergence-free.
- We fit the model to the data of SuperDARN, which is a radar network observing the ionospheric plasma velocity.
- The gaps of the data coverage of SuperDARN are filled with the empirical model (Weimber 2001).
- We assume the weight $\boldsymbol{w}$ can be decomposed into the model-based value $\boldsymbol{\zeta}$ and the residual $\boldsymbol{\beta}$:

  $$\boldsymbol{w} = \boldsymbol{\zeta} + \boldsymbol{\beta},$$

  and the residual $\boldsymbol{\beta}$ is estimated with the Kalman filter.
- We assume the temporal evolution of the weights $\boldsymbol{\beta}$

  $$p(\boldsymbol{\beta}_k \mid \boldsymbol{\beta}_{k-1}) = \mathcal{N}(\alpha \boldsymbol{\beta}_{k-1}, \mathrm{Q}).$$
- The residual component can then be estimated with the following Kalman filter algorithm:

  Prediction:

  $$\boldsymbol{\beta}_{k|k-1} = \alpha \boldsymbol{\beta}_{k-1|k-1},$$

  $$\mathrm{P}_{k|k-1} = \alpha^2 \mathrm{P}_{k-1|k-1} + \mathrm{Q}.$$

  Filtering:

  $$\boldsymbol{\beta}_{k|k} = \boldsymbol{\beta}_{k|k-1} + (\mathrm{P}_{k|k-1}^{-1} + \mathrm{H}_k^T \mathrm{R}_k^{-1} \mathrm{H}_k)^{-1} \mathrm{H}_k^T \mathrm{R}_k^{-1} (\boldsymbol{y}_k - \mathrm{H}_k \boldsymbol{\beta}_k),$$

  $$\mathrm{P}_{k|k} = (\mathrm{P}_{k|k-1}^{-1} + \mathrm{H}_k^T \mathrm{R}_k^{-1} \mathrm{H}_k)^{-1}.$$

Design of covariance matrices

- In order to ensure spatial smoothness, the covariance matrix Q_k is set as follows:

  $$\mathrm{Q} = \sigma_Q^2 \begin{pmatrix} C_Q(\boldsymbol{r}_1, \boldsymbol{r}_1) & \cdots & C_Q(\boldsymbol{r}_1, \boldsymbol{r}_n) \\ \vdots & \ddots & \vdots \\ C_Q(\boldsymbol{r}_n, \boldsymbol{r}_1) & \cdots & C_Q(\boldsymbol{r}_n, \boldsymbol{r}_n) \end{pmatrix},$$

  where $C_Q(\boldsymbol{r}_i, \boldsymbol{r}_j)$ is the following correlation function:

  $$C_Q(\boldsymbol{r}_i, \boldsymbol{r}_j) = \rho(\boldsymbol{r}_i, \boldsymbol{r}_j) \exp\left[\kappa \left(\frac{\boldsymbol{r} \cdot \boldsymbol{r}_i}{R^2} - 1 \right) \right],$$

  where $\rho(\boldsymbol{r}_i, \boldsymbol{r}_j) = \dfrac{(\sin\lambda_i - \sin 40^\circ)(\sin\lambda_j - \sin 40^\circ)}{(1 - \sin 40^\circ)^2}.$
- The matrix R_k is set as follows:

  $$\mathrm{R}_k = \sigma_R^2 \begin{pmatrix} \delta_{b_1 b_1} C_R(g_1, g_1) & \cdots & \delta_{b_1 b_1} C_R(g_1, g_1) \\ \vdots & \ddots & \vdots \\ \delta_{b_1 b_1} C_R(g_1, g_1) & \cdots & \delta_{b_1 b_1} C_R(g_1, g_1) \end{pmatrix},$$

  where b_i and g_i denote the beam number and the range gate of the i-th element of the observation $\boldsymbol{y}$ and

  $$C_R(g_i, g_j) = \exp\left[-\frac{(g_i - g_j)^2}{2} \right].$$

Results >> See our paper for detail (https://doi.org/10.1186/s40623-020-01168-4).