2014年6月13日　統計数理研究所　オープンハウス

確率的最大不等式，狭義可算性，および関連する話題

西山　陽一　　数理・推論研究系　准教授

1. A stochastic maximal inequality

• The most important special case of the Doob-Meyer decomposition equation for **1-dimensional** martingale difference sequence $(\xi_k)_{k=1,2,...}$ is:

$$\left(\sum_{k=1}^{n}\xi_k\right)^2 = \sum_{k=1}^{n}E[\xi_k^2|\mathcal{F}_{k-1}] + M_n.$$

• Our *stochastic maximal inequality* gives an inequality analogue to the Doob-Meyer decomposition for maxima of **finite** number of martingale difference sequences $(\xi_k^i)_{k=1,2,...}$, $i\in\mathbb{I}_F$, given by

$$\max_{i\in\mathbb{I}_F}\left(\sum_{k=1}^{n}\xi_k^i\right)^2 \wedge K \leq \frac{K}{1-e^{-K}}\left\{\sum_{k=1}^{n}E\left[\max_{i\in\mathbb{I}_F}(\xi_k^i)^2\middle|\mathcal{F}_{k-1}\right]+M_n\right\}.$$

[Key points of the proof]

• Let $X=(X^1,...,X^d)$ be a d-dimensional semimartingale.

Itô's inequality. If $f\in C^2$ and it is concave, then it holds that

$$f(X_t)-f(X_0) \leq \sum_{i=1}^{d}\int_0^t D_if(X_{s-})dX_s^i + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^t D_{ij}f(X_{s-})d\langle X^{c,i},X^{c,j}\rangle_s.$$

• Put $X_t^i := \sum_{k\leq t}\xi_k^i$ and define Y^i's by

$$Y_t^1 = 1\left\{|X_t^1| \geq \max_{1<j\leq m}|X_t^j|\right\},$$
$$Y_t^i = 1\left\{|X_t^i| > \max_{1\leq j<i}|X_t^j|,\ |X_t^i|\geq\max_{i<j\leq m}|X_t^j|\right\},\quad i=2,...,m-1,$$
$$Y_t^m = 1\left\{|X_t^m| > \max_{1\leq j<m}|X_t^j|\right\}.$$

• Then it holds that

$$\max_{1\leq i\leq m}\left(\sum_{k\leq t}\xi_k^i\right)^2 = \max_{1\leq i\leq m}(X_t^i)^2 = \sum_{i=1}^{m}(X_t^i)^2Y_t^i.$$

• Applying Itô's inequality to $f(\widetilde{x}_1,...,\widetilde{x}_m,y_1,...,y_m)=\psi\left(\sum_{i=1}^m\widetilde{x}_iy_i\right)$, we have

$$\begin{aligned}\psi&\left(\sum_{i=1}^{m}(X_t^i)^2Y_t^i\right)\\ &\leq \sum_{i=1}^{m}\int_0^t\psi'(Z_{s-})Y_{s-}^id(X_s^i)^2+\sum_{i=1}^{m}\int_0^t\psi'(Z_{s-})(X_{s-}^i)^2dY_s^i\\ &\leq \sum_{i=1}^{m}\int_0^t\psi'(Z_{s-})Y_{s-}^id(X_s^i)^2\\ &= \sum_{i=1}^{m}\int_0^t\psi'(Z_{s-})Y_{s-}^id\langle X^i\rangle_s+M_t,\end{aligned}$$

where $Z=(X^2,Y)$, M is a **local martingale** starting from zero and

$$\langle X^i\rangle_t=\sum_{k\leq t}E[(\xi_k^i)^2|\mathcal{F}_{k-1}].$$

2. Strict countability

Under which condition on the set $\mathbb{I}$ does the following "monotone convergence argument" hold true?

$$\lim_{m\to\infty}E\left[\max_{i\in\mathbb{I}_m}|X_i|\right]=E\left[\lim_{m\to\infty}\max_{i\in\mathbb{I}_m}|X_i|\right]=E\left[\sup_{i\in\mathbb{I}}|X_i|\right].$$

2.1. Hint from discussion to define "separability"

[A] Ledoux and Talagrand (1991) used the following definition:

$$E^*\left[\sup_{h\in\mathcal{H}}X(h)\right] := \sup_{F\subset\mathcal{H}}E\left[\max_{h\in F}X(h)\right],$$

where the $\sup_{F\subset\mathcal{H}}$ is taken over all **finite** subsets F of $\mathcal{H}$.

[B] Ledoux and Talagrand (1991) gives also the definition of *separability* of random field; there exists a negligible set N and a countable set $\mathcal{H}^*\subset\mathcal{H}$ such that, for every $\omega\in N^c$, every $h\in\mathcal{H}$ and $\varepsilon>0$,

$$X(h,\omega)\in\overline{\{X(\widetilde{h},\omega);\widetilde{h}\in\mathcal{H}^*,\rho(h,\widetilde{h})<\varepsilon\}},$$

and in this case, we can compute as

$$E\left[\sup_{h\in\mathcal{H}}X(h)\right]=E\left[\sup_{h\in\mathcal{H}^*}X(h)\right].$$

[D-1953] However, in Doob's (1953) original definition of separability, the dense subset $T^*\subset T$ is taken to be **not** a **countable set** but a **"sequence"**.

[D-1984] After three decades later, Doob (1984) again **suggested** how to define the concept of "separability" based on **"cofinal sequence"**.

[D-2004] However, Joseph L. Doob did not explicitely write the definition of the word **"sequence"**.

2.2. Definitions and facts

[Definitions]

• A well-ordering $<$ for a set $\mathbb{I}$ is called *σ-ordering* if it satisfies that $\#\langle i\rangle<\infty$ for every $i\in\mathbb{I}$, where $\langle i\rangle:=\{j\in\mathbb{I};j<i\}$.

• A σ-ordered set $(\mathbb{I},<)$ is called a *sequence*.

• A set $\mathbb{I}$ is said to be a *pre-sequence* or *strictly countable* if it is possible to assign a σ-ordering "$<$" to $\mathbb{I}$.

• A random feild $\{X(h);h\in\mathcal{H}\}$ indexed by a semimetric space $(\mathcal{H},\rho)$ is said to be *strictly separable* if there exists a negligible set N and a strictly countable set $\mathcal{H}^*\subset\mathcal{H}$ such that, for every $\omega\in N^c$, every $h\in\mathcal{H}$ and $\varepsilon>0$,

$$X(h,\omega)\in\overline{\{X(\widetilde{h},\omega);\widetilde{h}\in\mathcal{H}^*,\rho(h,\widetilde{h})<\varepsilon\}}.$$

[Facts]

• About $\mathbb{N}$:

[A1] $(\mathbb{N},<)$, where "$<$" is the usual ordering for $\mathbb{N}$, is a sequence.

[A2] $(\mathbb{N},<^b)$, where "$<^b$" is a "bad" well-ordering for $\mathbb{N}$, may not be a sequence.

[A3] $\mathbb{N}$, with no ordering, is a pre-sequnce (i.e., a strictly countable set).

• Proprieties we actually use:

[B] For any given pre-sequence $\mathbb{I}$, it holds for any σ-ordering and any mapping $x:\mathbb{I}\to\mathbb{R}$,

$$\lim_{m\to\infty}\max_{1\leq n\leq m}x(i_n)=\sup_{i\in\mathbb{I}}x(i),$$

where "i_n, $n\in\mathbb{N}$" denotes the corresponding **"natural numbering"**.

• Properties on union operations:

[C1] If each $\mathbb{I}^{(k)}$ is strictly countable, then $\bigcup_{k=1}^d\mathbb{I}^{(k)}$ is strictly countable.

[C2] The above is not true if $d=\infty$. Actually, $\mathbb{N}\times\mathbb{N}$ is **not** strictly countable.

[C3] **Any** set which can be expressed in the form of an infinite disjoint unifon of infinite sets is **not** strictly coutable.

[C4] Even the limit of increasing sequence of finite sets **may not** be strictly countable in general.