On a Theorem Concerning the Sums of Positive Independent Random Variables.

By Kyonori Kunisawa.

§1. Let $\{X_k \mid k=1, 2, \cdots\}$ be a sequence of positive independent random variables. We call the sums $S_n = \sum_{k=1}^{n} X_k$ to be relatively stable, if there exist $\{B_n > 0 \mid n=1, 2, \cdots\}$ such that for any $\varepsilon > 0$

$$(1) \qquad P\left(\left|\frac{S_n}{B_n} - 1\right| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0$$

We call further the random variables X_n to be relatively smale, if for any $\varepsilon > 0$

$$(2) \qquad P\left(\frac{X_k}{B_n} > \varepsilon\right) \xrightarrow[n\to\infty]{} 0$$

uniformly in regard to $1 \leq k \leq n$.

A. Khintchine[1] firstly showed a necessary and sufficient condition for the relative stability of S_n, when the chance variables X_n have a same distribution function. Next A. Bobroff[2] generaized this problem and gived a criterion when $\{X_n \mid n=1, 2, \cdots\}$ have not necessary a same distribution function.

1) A. Khintchine
Giorn. dell.' Ist It. d. Attuari 7 (1936) pp. 365 – 377.

2) A. Bobroff, On the relative stability of the sums of positive random variables (Russian). Scientific reports of Moscou national university. Math. 1939, pp. 191 - 202.

In this paper, applying the method of mean concentration function[3], we ~~shall~~ give a simple proof of another criterion of this problem.

§ 2. Denote by $\{F_n(x) \mid n=1,2,\cdots\}$ the distribution functions of $\{X_n \mid n=1,2,\cdots\}$, then we have the following theorem.

Theorem 1 The necessary and sufficient condition that the sums S_n $(n=1,2,\cdots)$ should be relatively stable and the chance variables X_n $(n=1,2,\cdots)$ should be relatively smale is that there exists a sequence of positive numbers $\{C_n > 0 \mid n=1,2,\cdots\}$ satisfying

$$\sum_{k=1}^{n}\{1-F_k(C_n)\} \xrightarrow[n\to\infty]{} 0, \quad \sum_{k=1}^{n}\int_0^\infty \frac{x C_n}{C_n^2+x^2}\, dF_k(x) \xrightarrow[n\to\infty]{} \infty.$$

Proof. Our condition is sufficient. Put

$$B_n = \sum_{k=1}^{n}\int_0^{C_n} x\, dF_k(x),$$

then

$$\frac{B_n}{C_n} = \sum_{k=1}^{n}\int_0^{C_n}\frac{x}{C_n}\, dF_k(x) = \sum_{k=1}^{n}\int_0^{C_n}\frac{x C_n}{C_n^2}\, dF_k(x)$$

$$\geq \sum_{k=1}^{n}\int_0^{C_n}\frac{x C_n}{C_n^2+x^2}\, dF_k(x) \xrightarrow[n\to\infty]{} \infty$$

Hence for any $\varepsilon > 0$, if n is sufficient large, we have $C_n \leq B_n \varepsilon$.

3) K. Kunisawa, Mean concentration function and the law of large numbers, Proc. Imp. Acad. Tokyo, 20 (1944), pp. 627-630.

Whence

$$P(X_k \geq \varepsilon B_n) \leq P(X_k \geq C_n) \leq \sum_{k=1}^{n}(1-F_k(C_n)) \to 0.$$

This inequality shows that the chance variables X_k are relatively small. Now put

$$f_k(t) = \int_{-\infty}^{\infty} e^{itx}\, dF_k(x), \qquad k = 1, 2, \cdots$$

then for $|t| \leq T$

$$\left|\sum_{k=1}^{n}(f_k(\tfrac{t}{B_n}) - 1) - it\right| = \left|\sum_{k=1}^{n}\left(f_k(\tfrac{t}{B_n}) - 1 - it\int_0^{C_n}\tfrac{x}{B_n}\, dF_k(x)\right)\right|$$

$$= \left|\sum_{k=1}^{n}\left(\int_0^{\infty}(e^{\frac{itx}{B_n}}-1)\, dF_k(x) - it\int_0^{C_n}\frac{x}{B_n}\, dF_k(x)\right)\right|$$

$$\leq \sum_{k=1}^{n}\left|\int_0^{C_n}(e^{\frac{itx}{B_n}} - 1 - \frac{itx}{B_n})\, dF_k(x)\right| + \sum_{k=1}^{n}\left|\int_{C_n}^{\infty}(e^{\frac{itx}{B_n}}-1)\, dF_k(x)\right|$$

$$\leq \frac{C_n}{2B_n}T^2 + 2\sum_{k=1}^{n}(1-F_k(C_n)) = o(T^2) + o(1).$$

As (3) implies $f_k(\frac{t}{B_n}) \xrightarrow[n\to\infty]{} 1$ uniformly for any finite interval $|t| \leq T$ and $k = 1, 2, \cdots, n$, we have for the interval $|t| \leq T$

$$\log \prod_{k=1}^{n} f_k(\tfrac{t}{B_n}) = \sum_{k=1}^{n} \log f_k(\tfrac{t}{B_n})$$

$$= -\sum_{k=1}^{n}\left\{(1-f_k(\tfrac{t}{B_n})) + \tfrac{1}{2}(1-f_k(\tfrac{t}{B_n}))^2 + \cdots\right\}$$

$$= \sum_{k=1}^{n}(f_k(\tfrac{t}{B_n}) - 1) + o(T) + o(T^2) + o(1)$$

$$= it + o(T) + o(T^2) + o(1)$$

Therefore we can conclude

$$\prod_{k=1}^{n} f_k(\tfrac{t}{B_n}) \xrightarrow[n\to\infty]{} e^{it}$$

uniformly for $|t| \leq T$. This fact shows the relative stability of S_n

Our condition is necessary. We assume that

$$\prod_{k=1}^{n} f_k(\tfrac{t}{B_n}) \longrightarrow e^{it}$$

uniformly for $|t| \leq T$ and $k = 1, 2, \cdots, n$. Hence we see

(5) $$\prod_{k=1}^{n} \left| f_k\left(\frac{t}{B_n}\right) \right|^2 \xrightarrow[n\to\infty]{} 1$$

uniformly for $|t| \leq T$ and

(6) $$\left| f_k\left(\frac{t}{B_n}\right) \right|^2 \xrightarrow[n\to\infty]{} 1$$

uniformly for $|t| \leq T$ and $k = 1, 2, \cdots, n$. From (5) we have for sufficiently large n

$$\prod_{k=1}^{n} \left| f_k\left(\frac{t}{B_n}\right) \right|^2 \geq \frac{1}{2}$$

for $|t| \leq T$. Hence by the method of the mean concentration function[3)]

$$\eta \int_0^T e^{-\eta t}\left(1 - \prod_{k=1}^{n} \left| f_k\left(\frac{t}{B_n}\right)\right|^2\right) dt \geq \frac{1}{2}\, \eta \sum_{k=1}^{n} \int_0^T e^{-\eta t}\left(1 - \left| f_k\left(\frac{t}{B_n}\right)\right|^2\right) dt$$

Applying the inequality $(1 - |f_k(2^i t)|^2) \leq 4^i (1 - |f_k(t)|^2)$ (i is a positive integer), we have an absolute constant K such that

$$\eta \sum_{k=1}^{n} \int_0^T e^{-\eta t}\left(1 - \left| f_k\left(\frac{t}{B_n}\right)\right|^2\right) dt \geq K \sum_{k=1}^{n} \int_{-\infty}^{\infty} \frac{x^2}{\eta^2 B_n^2 + x^2}\, d\tilde{F}_k(x)$$

$$= K \sum_{k=1}^{n} \eta \int_0^{\infty} e^{-\eta t}\left(1 - \left| f_k\left(\frac{t}{B_n}\right)\right|^2\right) dt$$

where $\tilde{F}_k(x)$ is the symmetrized distribution of $F_k(x)$. From (5) we have

(7) $$\frac{1}{2} \sum_{k=1}^{n} \left\{1 - \tilde{F}_k(\eta B_n)\right\} \leq \sum_{k=1}^{n} \int_0^{\infty} \frac{x^2}{\eta^2 B_n^2 + x^2}\, d\tilde{F}_k(x) \xrightarrow[n\to\infty]{} 0$$

By (6) we have also for sufficiently large n

$$F_k(\eta B_n) \geq \frac{1}{2}$$

uniformly in regard to $1 \leq k \leq n$. Hence

$$2(1 - \tilde{F}_k(\eta B_n)) = P(|X_k - Y_k| \geq \eta B_n).$$

$$\geq P(X_k - Y_k \geq \eta B_n) \geq P(X_k \geq 2\eta B_n,\ Y_k < \eta B_n)$$
$$= \{1 - F_k(2\eta B_n)\} F_k(\eta B_n)$$

where Y_k is a random variable independent of X_k and has the same distribution funktion $F_k(x)$ cos X_k. From (7) we see

$$\sum_{k=1}^{n} (1 - F_k(\eta B_n)) \xrightarrow[n\to\infty]{} 0$$

If we select properly a positive sequence tending to zero $\{\eta_n \mid n = 1, 2, \cdots\}$ such that

$$(8) \qquad \sum_{k=1}^{n} (1 - F_k(\eta_n B_n)) \xrightarrow[n\to\infty]{} 0,$$

we see

$$\sum_{k=1}^{n} \int_0^\infty \frac{x \eta_n B_n}{\eta_n^2 B_n^2 + x^2} dF_k(x) \xrightarrow[n\to\infty]{} 0$$

Otherwise, there should exist subsequence

$$n, n_2, n_3, \cdots, n_i, \cdots \longrightarrow \infty$$

and a constant M satisfying

$$\sum_{k=1}^{n_i} \int_0^\infty \frac{x \eta_{n_i} B_{n_i}}{\eta_{n_i}^2 B_{n_i}^2 + x^2} dF_k(x) \leq M.$$

Since we have

$$\left.\begin{array}{l} \dfrac{1}{2\eta_{n_i} B_{n_i}} \displaystyle\sum_{k=1}^{n_i} \int_0^{\eta_{n_i} B_{n_i}} x\, dF_k(x) \\ \dfrac{1}{2\eta_{n_i}^2 B_{n_i}^2} \displaystyle\sum_{k=1}^{n_i} \int_0^{\eta_{n_i} B_{n_i}} x^2\, dF_k(x) \end{array}\right\} \leq \sum_{k=1}^{n_i} \int_0^{\eta_{n_i} B_{n_i}} \frac{x \eta_{n_i} B_{n_i}}{\eta_{n_i}^2 B_{n_i}^2 + x^2} dF_k(x) \leq$$

$$\leq \sum_{k=1}^{n_i} \int_0^\infty \frac{x \eta_{n_i} B_{n_i}}{\eta_{n_i}^2 B_{n_i}^2 + x^2} dF_k(x) \leq M,$$

we have

$$(9) \qquad \frac{1}{B_{n_i}} \sum_{k=1}^{n_i} \int^{\eta_{n_i} B_{n_i}} x\, dF_k(x) \leq 2\eta_{n_i} M,$$

$$(10) \qquad \frac{1}{B_{n_i}^2} \sum_{k=1}^{n_i} \int_0^{\eta_{n_i} B_{n_i}} x^2\, dF_k(x) \leq 2\eta_{n_i}^2 M.$$

Whence, putting $C_n = \eta_n B_n$, in the analogous way with (4)

$$\left|\sum_{k=1}^{n}\left(f_k\left(\frac{t}{B_n}\right)-1-it\int_0^{C_n}\frac{x}{B_n}\,dF_k(x)\right)\right|$$

$$\leq \sum_{k=1}^{n}\left|\int_0^{C_n}\left(e^{\frac{itx}{B_n}}-1-\frac{itx}{B_n}\right)dF_k(x)\right|+\sum_{k=1}^{n}\left|\int_{C_n}^{\infty}\left(e^{\frac{itx}{B_n}}-1\right)dF_k(x)\right|$$

$$\leq \sum_{k=1}^{n} t^2\int_0^{C_n}\frac{x^2}{2B_n^2}\,dF_k(x)+2\sum_{k=1}^{n}(1-F_k(C_n)).$$

From (8), (9) and (10), we have for $|t|\leq T$

$$\left|\sum_{k=1}^{n_i}\left(f_k\left(\frac{t}{B_n}\right)-1\right)\right|\leq |t|\sum_{k=1}^{n_i}\int_0^{C_{n_i}}\frac{x}{B_{n_i}}\,dF_k(x)+t^2\sum_{k=1}^{n_i}\int_0^{C_{n_i}}\frac{x^2}{2B_{n_i}^2}\,dF_k(x)$$

$$+2\sum_{k=1}^{n_i}(1-F_k(C_{n_i}))$$

$$\leq T\cdot 2\eta_{n_i}M+T^2\cdot\eta_{n_i}^2 M+o(1)$$

Hence for $|t|\leq T$

$$\sum_{k=1}^{n_i}\left(f_k\left(\frac{t}{B_n}\right)-1\right)=o(1)+O(T^2)+O(T).$$

Whence

$$\log\prod_{k=1}^{n_i} f_k\left(\frac{t}{B_n}\right)=\sum_{k=1}^{n_i}\log f_k\left(\frac{t}{B_n}\right)=\sum_{k=1}^{n_i}\left(f_k\left(\frac{t}{B_{n_i}}\right)-1\right)+$$

$$+O(T^2)+O(T)+o(1).$$

This fact is contrary to the first assumption:

$$\prod_{k=1}^{n} f_k\left(\frac{t}{B_n}\right)\xrightarrow[n\to\infty]{} e^{it}$$

Remark. Following three conditions are equivalent: for properly chosen $C_n>0$ ($C_n \xrightarrow[n\to\infty]{} \infty$).

1° (A. Bobroff)

a) $\sum_{k=1}^{n}(1-F_k(c_n)) \xrightarrow[n\to\infty]{} 0$

b) $\sum_{k=1}^{n}\frac{1}{c_n}\int_0^{c_n}(1-F_k(x))dx \xrightarrow[n\to\infty]{} \infty$

2° a) $\sum_{k=1}^{n}(1-F_k(c_n)) \xrightarrow[n\to\infty]{} 0$

b) $\sum_{k=1}^{n}\int_0^{\infty}\frac{c_n x}{c_n^2+x^2}dF_k(x) \xrightarrow[n\to\infty]{} \infty$

3° a) $\sum_{k=1}^{n}(1-F_k(c_n)) \xrightarrow[n\to\infty]{} 0,$

b) $\sum_{k=1}^{n}\frac{1}{c_n}\int_0^{c_n}x\,dF_k(x) \xrightarrow[n\to\infty]{} \infty$

In fact, 1° implies 2°.

$$\sum_{k=1}^{n}\int_0^{\infty}\frac{c_n x}{c_n^2+x^2}dF_k(x) = \sum_{k=1}^{n}\int_{c_n}^{\infty}\frac{c_n x}{c_n^2+x^2}dF_k(x) + \sum_{k=1}^{n}\int_0^{c_n}x\,dF_k(x)$$
$$\geq \sum_{k=1}^{n}\frac{1}{2}\int_0^{c_n}\frac{x}{c_n}dF_k(x).$$

By partial integration

(11) $$\sum_{k=1}^{n}\int_0^{c_n}(1-F_k(x))dx = \sum_{k=1}^{n}\left\{(1-F_k(c_n))+\frac{1}{c_n}\int_0^{c_n}x\,dF_k(x)\right\}$$

Hence

$$\sum_{k=1}^{n}\int_0^{\infty}\frac{c_n x}{c_n^2+x^2}dF_k(x) \xrightarrow[n\to\infty]{} \infty$$

2° implies 3°. For from the above equation, it is evident.

3° implies 1°. It is evident, for

$$\sum_{k=1}^{n}\frac{1}{2}\int_0^{c_n}\frac{x}{c_n}dF_k(x) = \sum_{k=1}^{n}\int_0^{c_n}\frac{c_n x}{2c_n^2}dF_k(x) \leq$$
$$\leq \sum_{k=1}^{n}\int_0^{c_n}\frac{c_n x}{c_n^2+x^2}dF_k(x) \leq \sum_{k=1}^{n}\int_0^{\infty}\frac{c_n x}{c_n^2+x^2}dF_k(x).$$