Mean concentration function ト Quasi mean concentration function. III

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第二章　Quasi mean concentration function.

§ 2.1. Quasi mean concentration function.

(1.1.5) ノ $|f(t)|^2$ ヲ $f(t)$ ノ real part $\mathcal{R}(f(t))$ デオキカヘルト $F(x)$ ノ quasi mean concentration function (q. m. c. f.) ヲ得ル：

$$\phi_F(h) = h\int_0^\infty e^{-ht}\mathcal{R}(f(t))dt$$
$$= \int_{-\infty}^{\infty}\frac{h^2}{h^2+x^2}dF(x), \quad h>0.$$

此ノ $\phi_F(h)$ ハ明カニ h ノ non-negative, non-decreasing 且 continuous function デアル。明カニ

$$\lim_{h\to\infty}\phi_F(h) = 1.$$

サテ次ノ lemma ヲ考ヘル。此レハ Lemma 1.1.1 ト同様ニシテ証明サレル。

Lemma 2.1.1. 如何ナル $T>0$ ニ対シテモ

(2.1.3) $$1-\phi_F(h) \leq C(T)\int_0^T(1-\mathcal{R}(f(\tfrac{t}{h}))dt$$

此処ニ $C(T)$ ハ T ノミニ関係スル const. デアル。

Theorem 2.1.1. ドンナ $h>0$ ニ対シテモ

(2.1.4) $$1-\phi_F(h) \geq \frac{1}{2}(1-\psi_F(h))$$

テ $0\le t\le T$ ニ対シ $\mathcal{R}(f(t))>\delta>0$ ナル様

T ガ存在スルナラバ

$$(2.1.5)\qquad 1-\psi_F(h)\geq R(T,h)(1-\phi_F(h))$$

此処ニ $R(T,h)$ ハ T ト h ノミニ関係スル

デアル。

<u>証明</u>　明カニ

$$1-|f(t)|^2=1-(\mathcal{R}(f(t)))^2-(\mathcal{J}(f(t)))^2$$
$$\leq 2(1-\mathcal{R}(f(t)))$$

デアルカラ (2.1.4) ハ明ラカ。

假定ヨリ $\mathcal{R}(f(t))>\delta>0$ $(0\le t\le T)$ デアルカラ $0\le t\le T$ ニ対シ

$$1-|f(t)|^2\geq 1-\mathcal{R}(f(t))-(\mathcal{J}(f(t)))^2$$
$$\geq(1-\mathcal{R}(f(t)))-\int_{-\infty}^{\infty}\sin^2 tx\,dF($$
$$=1-\mathcal{R}(f(t))-\frac{1}{2}\int_{-\infty}^{\infty}(1-\cos 2tx$$
$$=1-\mathcal{R}(f(t))-\frac{1}{2}(1-\mathcal{R}(f(2$$

故ニ

$$h\int_0^T e^{-ht}(1-|f(t)|^2)\,dt$$
$$\geq h\int_0^T e^{-ht}(1-\mathcal{R}(f(t)))\,dt-\frac{h}{2}\int_0^T e^{-}$$
$$\mathcal{R}(f(2t))\,dt\geq$$
$$\geq\int_0^{hT}\left(e^{-t}-\frac{1}{4}e^{-\frac{t}{2}}\right)\left(1-\mathcal{R}\left(f\left(\frac{t}{h}\right)\right)\right)dt$$

一方十分小ナル $\eta>0$ ニ対シ $T_0>0$ ヲ選ビ

$0 \leq t \leq T_0$ ニ対シ

$$e^{-t}-\frac{1}{4}e^{-\frac{t}{2}} \geq \eta > 0$$

従ッテ $T_1 = \min.\{T_0, hT\}$ ニ対シ

$$1-\psi_F(h) \geq h\int_0^T e^{-ht}(1-|f(t)|^2)\,dt$$

$$\geq \int_0^{hT}\left(e^{-t}-\frac{1}{4}e^{-\frac{t}{2}}\right)\left(1-\Re\left(f\left(\frac{t}{h}\right)\right)\right)dt$$

$$\geq \eta\int_0^{T_1}(1-\Re(f(t)))\,dt$$

Lemma 2.1.1 ヨリ

$$1-\psi_F(h) \geq \eta\int_0^{T_1}(1-\Re(f(t)))\,dt$$

$$\geq \frac{\eta}{C(T_1)}(1-p_F(h))$$

$\eta/C(T_1) = R(T,h)$ トオケバ

$$1-\psi_F(h) \geq R(T,h)(1-p_F(h)).$$

定理 2.1.2. $\{F_1(x), \cdots, F_n(x)\}$ ヲ分布函数ノ任意ノ集合トスル.

$$f_k'(t) \equiv \int_{-\infty}^{\infty} e^{itx}dF_k'(x) \equiv \int_{-\infty}^{\infty} e^{itx}dF_k\left(x+\int_{-A}^{A}x\,dF_k(x)\right),$$

$(A>0)$

トオクナラバ

$$\sum_{k=1}^{n}\left|f_k'\left(\frac{t}{A}\right)-1\right| \leq (t^2+2|t|+4)\sum_{k=1}^{n}(1-p_{F_k}(A))$$

証明

□ (gini)

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簡單ノタメ

$$b_k = \int_{-A}^{A} x\, dF_k(x), \quad (k = 1, 2, \cdots)$$

$$\sum_{k=1}^{n} \left| f_k'\left(\frac{t}{A}\right) - 1 \right| = \sum_{k=1}^{n} \left| \int_{-\infty}^{\infty} e^{\frac{itx}{A}} dF_k'(x) \right.$$

$$\doteqdot \sum_{k=1}^{n} \left| \int_{-\infty}^{\infty} \left(e^{it\frac{(x-b_k)}{A}} - 1 \right) dF_k(x) \right|$$

$$\leq \sum_{k=1}^{n} \left\{ \left| \int_{|x| \leq A} \left(e^{it\frac{(x-b_k)}{A}} - 1 \right) dF_k(x) \right| + \right.$$

$$\leq \sum_{k=1}^{n} \left| \int_{|x| \leq A} \left(e^{it\frac{(x-b_k)}{A}} - \frac{it(x-b_k)}{A} - 1 \right) d \right.$$

$$+ \sum_{k=1}^{n} |t| \left| \int_{|x| \leq A} \frac{x-b_k}{A} dF_k(x) \right\} + 2 \sum_{k=1}^{n} \int_{|x|}$$

$$\leq \sum_{k=1}^{n} \frac{t^2}{2} \int_{|x| \leq A} \frac{(x-b_k)^2}{A^2} dF_k(x) + \sum_{k=1}^{n} |t| \frac{|b_k|}{A}$$

$$+ 2 \sum_{k=1}^{n} \int_{|x| > A} dF_k(x)$$

$$\leq \sum_{k=1}^{n} \left\{ \frac{t^2}{2} \left(\int_{|x| \leq A} \frac{x^2}{A^2} dF_k(x) - \frac{2b_k^2}{A^2} + \frac{b_k^2}{A^2} \int_{|x|} \right. \right.$$

$$\left. + |t| \int_{|x| > A} dF_k(x) + 2 \int_{|x| > A} dF_k(x) \right\}$$

$$\leq \sum_{k=1}^{n} \left\{ \frac{t^2}{2} \int_{|x| \leq A} \frac{x^2}{A^2} dF_k(x) + |t| \int_{|x| > A} dF_k \right.$$

$$\left. 2 \int_{|x| > A} dF_k(x) \right\} \leq (t^2 + 2|t| + 4) \sum_{k=1}^{n} \int$$

$$= (t^2 + 2|t| + 4) \sum_{k=1}^{n} (1 - \phi_{F_k}(A))$$

以上ノ二ケノ定理ハ定理1.1.1ト1.1.2ト共ニ今後基本的ナ役割ヲモツ。

§2.2. The relation between g. m. c. f. and the convolution of probability distribution.

g. m. c. f. ハ必ズシモ probability distribution ノ convolution ニヨリ減少シナイ。例ヘバ

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 \leq x < 1 \\ 1 & 1 \leq x, \end{cases}$$

ヲ考ヘル。$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x) = \cos t$

故ニ

$$\phi_F(h) = h\int_0^\infty e^{-ht} \Re(f(t))dt = \frac{h^2}{h^2+1}$$

一方

$$\phi_{F*F}(h) = h\int_0^\infty e^{-ht} \Re(f^2(t))dt$$

$$= h\int_0^\infty e^{-ht} \left(\frac{\cos 2t + 1}{2}\right) dt$$

$$= \frac{1}{2}\left(\frac{h^2}{h^2+4} + 1\right)$$

故ニ $h < \sqrt{2}$ ニ対シ

$$\phi_F(h) < \phi_{F*F}(h)$$

然シ定理1.2.1ト中ノ相似ナ定理ヲ示

ス事ガ出來ル。

random variables ノ system (1.2.1) $\| X_{nm} \|$ ガ與ヘラレ、$F_{nm}(x)$ ヲ X_{nm} ノ分布函数トスル。$D_n(\alpha)$ ヲ次ノ様ニシテ定義サレル函数トスル。

$$(2.2.1) \qquad \alpha = \phi_{F_{n1} * \cdots \phi_{nm_n}}(D_n(\alpha)), \quad (1 > \alpha > 0)$$

<u>定理 2.2.1.</u> 二ケノ実数 α, β $(\frac{3}{4} < \alpha \leq 1,\ \frac{7}{8} < \beta \leq 1)$ ガアタヘラレ時次ノ関係ヲミタス正ノ consts. K ト N (共ニ α ト β ノミニ depend スル) ヲ定メル事ガ出來ル。即チ $m_n > N$, $D_n(\beta) \geq l_0$ 且ツ

$$(2.2.2) \qquad \phi_{F_{nm}}(l_0) = \alpha, \quad m = 1, 2, \ldots, m_n,$$

ガミタサレルナラバ

$$(2.2.3) \qquad D_n(\beta) \geq \sqrt{m_n}\, l_0 K, \quad (m_n \geq N)$$

従ッテ

$$(2.2.4) \qquad \phi_{F_{n1} * \cdots * F_{nm_n}}(\sqrt{m_n}\, l_0 K) \leq \beta.$$

<u>証明</u> 定理 2.1.1 ノ (2.1.4) ヨリ

$$(2.2.5) \qquad \begin{aligned} 1 - \beta &= \int_{-\infty}^{\infty} \frac{x^2}{D_n^2(\beta) + x^2}\, dF_{n1} * \cdots * F_{nm_n}(x) \\ &\geq \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{D_n^2(\beta) + x^2}\, d\tilde{F}_{n1} * \cdots * \tilde{F}_{nm_n}(x) \end{aligned}$$

$$\geq \frac{1}{4}\left\{\int_{|x|\leq D_n(\beta)} \frac{x^2}{D_n^2(\beta)} d\tilde{F}_{n1} * \cdots * \tilde{F}_{n1}(x)\right.$$

$$\left. + \int_{|x|>D_n(\beta)} d\tilde{F}_{n1} * \cdots * \tilde{F}_{nm_n}(x)\right.$$

一方

$$1 - \prod_{m=1}^{m_n} \left| f_{nm}\left(\frac{t}{D_n(\beta)}\right)\right|^2 = \int_{-\infty}^{\infty} \left(1 - \cos\frac{tx}{D_n(\beta)}\right) d\tilde{F}_{n1} * \cdots * \tilde{F}_{nm_n}(x)$$

$$= 2\int_{-\infty}^{\infty} \sin^2\left(\frac{tx}{2D_n(\beta)}\right) d\tilde{F}_{n1} * \cdots * \tilde{F}_{nm_n}(x) \leq$$

$$\leq 2\left\{ t^2 \int_{|x|\leq D_n(\beta)} \frac{x^2}{4D_n^2(\beta)} d\tilde{F}_{n1} * \cdots * \tilde{F}_{nm_n} + \right.$$

$$\left. + \int_{|x|>D_n(\beta)} d\tilde{F}_{n1} * \cdots * \tilde{F}_{nm_n}(x)\right.$$

$1 \geq \beta > \frac{7}{8}$ ト (2.2.5) ヨリ $0 \leq t \leq 2$ ニ対シ

$$1 - \prod_{m=1}^{m_n} \left| f_{nm}\left(\frac{t}{D_n(\beta)}\right)\right|^2 \leq 2\left\{ \int_{|x|\leq D_n(\beta)} \frac{x^2}{D_n^2(\beta)} d\tilde{F}_{n1} * \cdots * \tilde{F}_{nm_n}(x)\right.$$

$$\left. + \int_{|x|>D_n(\beta)} d\tilde{F}_{n1} * \cdots * \tilde{F}_{nm_n}(x)\right. \leq 8(1-\beta) = \delta' < 1$$

故ニ $0 \leq t \leq 2$ ニ対シテ

$$0 < \delta = 1 - \delta' \leq \prod_{m=1}^{m_n} \left| f_{nm}\left(\frac{t}{D_n(\beta)}\right)\right|^2 =$$

$$= \prod_{m=1}^{m_n} \left| \int_{-\infty}^{\infty} e^{itx} d\tilde{F}(D_n(\beta)x)\right|^2$$

定理 1.1.2 ヨリ

$$\sum_{m=1}^{m_n} (1 - \psi_{G_{nm}}(1)) \leq K(1 - \psi_{G_{n1} * \cdots * G_{nm}}(1))$$

但シ　K ハ abs. const.　$G_{nm}(x) \equiv F_{nm}(D_n(\beta)x)$

(2.2.5) ヨリ

$$\sum_{m=1}^{m_n}(1-\psi_{G_{nm}}(1)) = \sum_{m=1}^{m_n}\int_0^\infty e^{-t}\left(1-\left|f_{nm}\left(\frac{t}{D_n(\beta)}\right)\right|^2\right)dt$$

$$= \sum_{m=1}^{m_n}\int_{-\infty}^{\infty}\frac{x^2}{D_n^2(\beta)+x^2}\,d\tilde{F}_{nm}(x) \leq K\left(1-\psi_{G_{n1}*\cdots*G_{nm_n}}(1)\right)$$

$$= K\int_0^\infty e^{-t}\left(1-\prod_{m=1}^{m_n}\left|f_{nm}\left(\frac{t}{D_n(\beta)}\right)\right|^2\right)dt \leq 2(1-\beta)K$$

他方 (2.2.2) ト $D_n(\beta) \geq l_0$ ヨリ

$$1-\alpha = \int_{-\infty}^{\infty}\frac{x^2}{l_0^2+x^2}\,dF_{nm}(x) \geq \int_{-\infty}^{\infty}\frac{x^2}{D_n^2(\beta)+x^2}$$

(2.2.7)

$$\geq \frac{1}{2}\int_{|x|\leq D_n(\beta)} x^2/D_n^2(\beta)\,dF_{nm}(x) + \frac{1}{2}\int_{|x|>D_n(\beta)} dF_{nm}(x),$$

$m = 1, 2, \cdots, m_n$

然ルニ

$$1-\mathcal{R}\left(f_{nm}\left(\frac{t}{D_n(\beta)}\right)\right) = \int\left(1-\cos\frac{tx}{D_n(\beta)}\right)dF_{nm}(x)$$

$$= 2\int_{-\infty}^{\infty}\sin^2\frac{tx}{2D_n(\beta)}\,dF_{nm}(x)$$

$$\leq 2\left\{t^2\int_{|x|\leq D_n(\beta)} x^2/4D_n^2(\beta)\,dF_{nm}(x) + \int_{|x|>D_n(\beta)} dF_{nm}(x)\right\},$$

$m = 1, 2, \cdots, m_n,$

従ッテ (2.2.7) ヨリ　$0 \leq t \leq 2$ ニ対シ

$$1-\mathcal{R}\left(f_{nm}\left(\frac{t}{D_n(\beta)}\right)\right) \leq 4(1-\alpha),$$

$m = 1, 2, \cdots, m_n$

故ニ $3/4 < \alpha \leq 1$ カラ $0 \leq t \leq 2$ ニ対シ

$$0 < 1-4(1-\alpha) \leq \mathcal{R}\left(f_{nm}\left(\frac{t}{D_n(\beta)}\right)\right),$$

$$m = 1, 2, \cdots, m_n$$

定理 2.1.1 ノ (2.1.5) ヨリ

$$1-\psi_{G_{nm}}(1) \geq R(1-\phi_{G_{nm}}(1)) = R\int_{-\infty}^{\infty}\frac{x^2}{D_n^2(\beta)+x^2}dF_{nm}(x)$$

$$m=1, 2, \cdots, m_n$$

但シ R ハ abs. const. (2.2.6) ヨリ

$$\frac{2(1-\beta)K}{R} \geq \sum_{m=1}^{m_n}\int_{-\infty}^{\infty}\frac{x^2}{D_n^2(\beta)+x^2}dF_{nm}(x)$$

$$\geq \sum_{m=1}^{m_n}\frac{l_0^2}{D_n^2(\beta)+l_0^2}\left\{\int_{|x|\leq l_0} x^2/l_0^2\, dF_{nm}(x) + \right.$$

$$\left. + \int_{|x|>l_0} dF_{nm}(x)\right\} \geq \sum_{m=1}^{m_n}\frac{l_0^2}{D_n^2(\beta)+l_0^2}\int_{-\infty}^{\infty}\frac{x^2}{l_0^2+x^2}dF_{nm}(x)$$

(2.2.2) ヨリ

$$\frac{2(1-\beta)K}{R} \geq \sum_{m=1}^{m_n}\frac{l_0^2}{D_n^2(\beta)+l_0^2}\int_{-\infty}^{\infty}\frac{x^2}{l_0^2+x^2}dF_{nm}(x)$$

$$= \frac{m_n l_0^2}{D_n^2(\beta)+l_0^2}(1-\alpha)$$

故ニ

$$D_n(\beta) \geq \sqrt{m_n}\, l_0 \sqrt{\frac{(1-\alpha)R}{2(1-\beta)K} - \frac{1}{m_n}}$$

依ッテ $m_n > N(\alpha, \beta) \doteqdot \dfrac{2(1-\beta)K}{(1-\alpha)R}$ ナラバ

$$D_n(\beta) \geq \sqrt{m_n}\, l_0 K, \quad K = \sqrt{\frac{(1-\alpha)R}{2(1-\beta)K} - 1}$$

故ニ

$$\phi_{F_{n1} * \cdots * F_{nm_n}}(\sqrt{m_n}\, l_0 K) \leqq \beta$$

但シ K ハ α ト β ノミニ depend スル

（ツヾ