Representation of a function by the Fourier-Stieltjes integral. (II)

by Tatsuo Kawada, in Tokyo.

1. Let $\sigma(x)$ be a non-decreasing function such that $\sigma(-\infty)=0$, $\sigma(\infty)=1$ and let the class of function which can be represented as

$$(1.1) \qquad f(t)=\int_{-\infty}^{\infty} e^{itx}\, d\sigma(x).$$

be K_1. And denote the class of functions which can be represented as

$$(1.2) \qquad f(t)=\int_{-\infty}^{\infty} e^{itx}\, dF(x),$$

where $F(x)$ be a function of bounded variation in $(-\infty,\infty)$, by K_2. In the former paper with same title[(1)] we have given the characterisations of classes K_1 and K_2. We give here another characterisation and we deduce known results from it. In (1) the Fourier series of certain functions has played an essential rôle, and in this paper we use the Fourier transform instead of it.

2. The theorems we shall prove are the followings.

Theorem 1. Let $f(t)$ be a continuous function such that $f(0)=1$ and there exists an R_0 such that for every $R \geqq R_0$.

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(1) T. Kawada, Representation of a function by Fourier-Stieltjes integral (I). Under the press.

Fourier transform

$$F(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(u) e^{iut} du.$$

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(2.1) $\int_{-R}^{R} f(t) e^{-ixt} dt \geq 0$, for every x,

or let $f(t)$ be a limit function to which

and ? certain sequence of such functions converges uniformly in every finite interval. Then

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$f(t)$ can be represented as (1.1) with some $G(x)$.

Conversely, for the function $f(t)$ of the form (1.1), either there exists an R_0 such that for every $R \geq R_0$, (2.1) holds, or it is a limit function to which certain sequence of such functions tends uniformly in every finite interval.

Theorem 2. Let $f(t)$ be a bounded continuous function such that there exists an R_0 such that for every $R \geq R_0$,

(2.2) $\int_{-\infty}^{\infty} dx \left| \int_{-R}^{R} f(t) e^{-ixt} dt \right| < M$,

M being a constant independent of R, or let $f(t)$ be a limit function of certain sequence of such functions in $(-\infty, \infty)$. Then $f(t)$ can be represented as (1.2) with $\int |dF(x)| < \infty$. Conversely for the function $f(t)$ of the form (1.2) with $\int |dF(x)| < \infty$, either there exists an R_0 such that for every $R \geq R_0$, (2.2) holds, or it is a limit function of certain sequence of such functions.

We shall prove these theorems. First we shall show the former part of Theorem 1.

Put

$$(2.3)\qquad f_R(t) = f(t), \qquad -R \leqq t \leqq R,$$
$$\qquad\qquad = 0 \qquad\qquad |t| \geqq R.$$

Then by the wellknown theorem on Fourier integral,

$\frac{1}{2}\{f(x+0)+f(x-0)\}$

$$(2.4)\qquad f_R(x) = \lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A}\left(1-\frac{|u|}{A}\right)e^{ixu}\,du\int_{-\infty}^{\infty}f_R(t)\,e^{-iut}\,dt$$

holds at continuity points of $f_R(x)$. Since by (2.1) $\int_{-\infty}^{\infty} f_R(t)e^{-itx}\,dt = \int_{-R}^{R} f(t)\,e^{-iut}\,dt \geqq 0$, it holds instead of (2.4), that

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$$f_R(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixu}\,du\int_{-\infty}^{\infty}f_R(t)\,e^{-iut}\,dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_R(u)\,e^{ixu}\,du$$

say. Since $\varphi(u) \geqq 0$, if we put $\frac{1}{2\pi}\int_{-\infty}^{u}\varphi_R(u)\,du = \sigma_R(u)$, then $\sigma_R(u)$ is a non-decreasing function such that $\sigma_R(-\infty)=0$, and $\sigma_R(\infty)=1$, for $\sigma_R(\infty) = f_R(0) = f(0) = 1$.

Thus we have

$1=0$ H4 $\sigma_R(\infty) = ?$

$$(2.5)\qquad f_R(x) = \int_{-\infty}^{\infty} e^{ixu}\,d\sigma_R(u)\ \text{[1]}$$

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(1) Since the right hand side is continuous in $(-\infty, \infty)$, if (2.1) holds, $f(t)$ must vanish for large $|t|$.

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Further since $\lim_{R\to\infty} f_R(t) = f(t)$ holds uniformly in every finite interval, by the Lévy continuity theorem[1], there exists $\sigma(x)$ such that $f(t)$ can be also written as (1.1).

Next we prove the converse. Let

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\, d\sigma(x),$$

$\sigma(\infty) = 1$, $\sigma(-\infty) = 0$ and $\sigma(x)$ being non-decreasing.

We put [illegible]

$$\varphi_A(t) = \begin{cases} 1 - \dfrac{|t|}{A}, & \text{for } |t| \leqq A, \\ 0, & \text{for } |t| > A. \end{cases}$$

Then

$$\varphi_A(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(Au/2)}{Au^2/2} \cos ut\, du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(Au/2)}{Au^2/2} e^{iut}\, du$$

$$= \int_{-\infty}^{\infty} e^{iut}\, d\sigma_A(u)$$

where $\sigma_A(u) = \frac{1}{\pi}\int_{-\infty}^{u} \frac{\sin^2(Au/2)}{Au^2/2} du$ and $\sigma_A(-\infty) = 0$ and $\sigma_A(\infty) = 1$. Clearly we can write as

$$\varphi_A(t) f(t) = \int_{-\infty}^{\infty} e^{ixt} d\left(\int_{-\infty}^{\infty} \sigma_A(x-u)\, d\sigma(u)\right).$$

This is of the form (1.1) and

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(1) For example see H. Cramér, Random variables and probability distributions, Cambr. Tracts. 1937. Theorem 11. P. 29

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$$\lim_{A\to\infty} \varphi_A(t) f(t) = f(t)$$

holds uniformly in every finite interval. Thus by the Lévy theorem mentioned, it suffices to prove that for $R \geqq A$,

$$(2.6) \quad \int_{-R}^{R} e^{-ixt} \varphi_A(t) f(t)\, dt \geqq 0, \qquad -\infty < x < \infty$$

But

$$\int_{-R}^{R} e^{-ixt} \varphi_A(t) f(t)\, dt = \int_{-A}^{A} e^{-ixt} \left(1 - \frac{|t|}{A}\right) f(t)\, dt$$

$$= \int_{-A}^{A} e^{-ixt} \left(1 - \frac{|t|}{A}\right) dt \int_{-\infty}^{\infty} e^{iut}\, d\sigma(u)$$

$$= \int_{-\infty}^{\infty} d\sigma(u) \int_{-A}^{A} \left(1 - \frac{|t|}{A}\right) e^{-it(x-u)}\, du$$

$$= 2\int_{-\infty}^{\infty} \frac{\sin^2(A(x-u)/2)}{A(x-u)^2/2}\, d\sigma(u)$$

$$\geqq 0$$

which proves the required.

The proof of Theorem 2 can also be done similarly. We use the similar notations. Let $f(t)$ be a continuous bounded function satisfying (2.2). Then as in the proof of Theorem 1, we have

$$f_R(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixu}\, du \int_{-\infty}^{\infty} f_R(t) e^{-iut}\, dt,$$

which is written as

$$f_R(x) = \int_{-\infty}^{\infty} e^{ixu}\, d F_R(u),$$

where

$$F_R(u) = \frac{1}{2\pi}\int_{-\infty}^{u} du \int_{-\infty}^{\infty} f_R(t)e^{-iut}\,dt.$$

Clearly we have

$$\int_{-\infty}^{\infty} |dF_R(u)| = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| \int_{-R}^{R} f(t)e^{-iut}dt \right| du < M$$

Thus $f_R(x)$ is of the form (1.2). The remainings of the former part can be done by the similar reasoning as in the proof of Theorem 1.

Next we consider the converse. As in the proof of Theorem 1, we have, for $R \geq A$,

$$\int_{-\infty}^{\infty} \left| \int_{-R}^{R} e^{-ixt}\varphi_A(t)f(t)\,dt \right| dx$$

$$= 2\int_{-\infty}^{\infty} dx \left| \int_{-\infty}^{\infty} \frac{\sin^2 A(x-u)/2}{A(x-u)^2/2}\,dF(u) \right|$$

$$= 2\int_{-\infty}^{\infty} |dF(u)| \int_{-\infty}^{\infty} \frac{\sin^2 A(x-u)}{A(x-u)^2/2}\,dx.$$

$$= \int_{-\infty}^{\infty} |dF(u)|,$$

from which by analogous arguments we can complete the proof of our Theorem [1].

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[1] In the proof of Theorem 2, we used, instead of Levy theorem mentioned, the analogous theorem concerning the sequence of functions of the form (1.2). See for example, S. Bochner, Fourier-sche Integrale, 1937, Leipzig. Satz 20, p. 70.

3. We shall apply Theorems 1 and 2 to prove some known results.

Theorem 3. (S. Bochner)[2]. If $f(t)$ is continuous, $f(0)=1$ and for any function $g(x)$ vanishing outside the finite interval, it holds that

$$(3.1) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\, g(x)\, \bar{g}(y)\, dx\, dy \geqq 0,$$

than $f(t)$ can be represented as (1.1). And the converse is also true.

The converse is almost evident putting (1.1) into (3.1) and so we prove the formet part only.

Let $g(x)$ be $(A)^{-\frac{1}{2}} e^{-ixt}$ in $0 \leqq x \leqq A$, and vanish elsewhere. Then by (3.1), we have

$$0 \leqq \frac{1}{A}\int_0^A \int_0^A f(x-y)\, e^{-i(x-y)t}\, dx\, dy$$

$$= \frac{1}{A}\int_0^A dy \int_{-y}^{A-y} f(u)\, e^{-iut}\, du$$

$$= \int_{-A}^{A} \left(1 - \frac{|u|}{A}\right) f(u)\, e^{-iut}\, du.$$

Thus if we consider the function

$$f(t, A) = f(t)\left(1 - \frac{|t|}{A}\right), \quad \text{for } |t| \leqq A$$
$$= 0, \qquad \text{for } |t| > A,$$

(1) S. Bochner. loc. cit..

then for every $R \geqq A$,

$$\int_{-R}^{R} f(t, A)\, e^{-ixt}\, dt \geqq 0.$$

And $f(t, A)$ converges $f(t)$ uniformly in every finite interval as $A \to \infty$. Hence Therem 1 shows our result.

Next we shall prove the Khintchine's theorem.

Theorem 4 (A. Khintchine)(1) The function of the form

$$(3.2) \qquad f(t) = \frac{1}{C}\int_{-\infty}^{\infty} \psi(x+t)\overline{\psi(x)}\, dx,$$

where $\psi(x) \in L_2(-\infty, \infty)$ and $C = (\int_{-\infty}^{\infty} |\psi(x)|^2\, dx)$ or the uniform limit function in every finite interval of a sequence of such functions can be represented as (1.1). And the converse is also true.

We prove the former part. By Lévy theorem, it is sufficient to prove that the function (3.2) can be written as (1.1), with $f(0) = 1$.

Now we define the function

$$\psi_A(x) = \psi(x), \quad |x| \leqq A,$$
$$\qquad = 0, \quad |x| > A.$$

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(1) A. Khintchine, Zur Kennzeichnung der characteristische Funktion. Bull. de l'univ. d'état à Moscou, vol. I (1937).

Then

$$(3.3) \qquad \lim_{A\to\infty}\int_{-\infty}^{\infty}\psi_A(x+t)\overline{\psi_A(x)}\,dx = \int_{-\infty}^{\infty}\psi(x+t)\overline{\psi(x)}\,dx$$

holds uniformly in $-\infty < t < \infty$. For Because!

$$\left|\int_{-\infty}^{\infty}\psi_A(x+t)\overline{\psi_A(x)}\,dx - \int_{-\infty}^{\infty}\psi(x+t)\overline{\psi(x)}\,dx\right|$$

$$\leq \left|\int_{-\infty}^{\infty}\psi_A(x+t)\{\overline{\psi_A(x)}-\overline{\psi(x)}\}\,dx\right| + \left|\int_{-\infty}^{\infty}\overline{\psi(x)}\{\psi_A(x+t)-\psi(x+t)\}\,dx\right|$$

which does not exceed by Schwarz's inequality,

$$\left(\int_{-\infty}^{\infty}|\psi_A(x+t)|^2dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}|\psi_A(x)-\psi(x)|^2dx\right)^{\frac{1}{2}} + \left(\int_{-\infty}^{\infty}|\psi(x)|^2dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}|\psi_A(x+t)-\psi(x+t)|^2dx\right)^{\frac{1}{2}}$$

$$\leq 2\left(\int_{-\infty}^{\infty}|\psi(x)|^2dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}|\psi_A(x)-\psi(x)|^2dx\right)^{\frac{1}{2}},$$

which does not containt and converges to zero as $A\to\infty$. Hence (3.3) holds uniformly in $(-\infty < t < \infty)$.

Thus it suffices to show that $\varphi_A(x) = (C_A)^{-1}\int_{-\infty}^{\infty}\psi_A(x+t)\overline{\psi_A(x)}\,dx$ can be written as (1.1) and $\varphi_A(0) = 1$.

$\varphi_A(0)=1$ is evident. Now take any number R greater than $2A$. Then

$$\int_{-R}^{R}\varphi_A(t)e^{-ixt}dt = \int_{-R}^{R}e^{-ixt}dt\,\frac{1}{C_A}\int_{-\infty}^{\infty}\psi_A(u+t)\overline{\psi_A(u)}\,du$$

$$= \frac{1}{C_A}\int_{-A}^{A} \overline{\varphi_A(u)}\, e^{iux} du \int_{-R}^{R} \varphi_A(u+t)\, e^{-ix(t+u)} dt$$

$$= \frac{1}{C_A}\int_{-\infty}^{\infty} \varphi_A(u)\, e^{iux} du \int_{-\infty}^{\infty} \varphi_A(u+t)\, e^{-ix(t+u)} dt,$$

noticing $|u+t| > A$ for $|u| < A$, $|t| > R$, which is

$$\frac{1}{C_A}\left|\int_{-\infty}^{\infty} \varphi(u)\, e^{-iux} du\right|^2 \geq 0.$$

Theorem 1 shows our required.

Next we consider the converse. Let $f(t)$ be of the form (1.1) with $f(0)=1$; then since this is, by Theorem 1, a uniform limit function in every finite interval of functions for which (2.1) holds. Thus for our purpose, it is sufficient to show that a function satisfying (2.1) is a uniform limit functions in every finite interval which is of the form (3.2). Moreover since $f_R(t) = f(t)$ ($|t| \leq R$), $= 0$ ($|t| > R$) converges uniformly in every finite interval to $f(t)$ as $R \to \infty$, it suffices to show that $f_R(t)$ can be written as (3.2).

As in the proof of Theorem 1, we have, for $R > R_0$,

$$(3.4) \qquad f_R(x) = \int_{-\infty}^{\infty} e^{-itx} dt \int_{-\infty}^{\infty} f_R(u)\, e^{itu} du.$$

If we put $\varphi_R(t) = \left(\int_{-\infty}^{\infty} f_R(u)\, e^{itu} du\right)^{1/2}$, the inner expression being non-negative by assumption, then clearly $\varphi_R(t) \in L_2(-\infty, \infty)$ and (3.4) is written as

$$f_R(x) = \int_{-\infty}^{\infty} \varphi_R^2(t) e^{-ixt}\, dt.$$

Thus by Parseval theorem

$$\int_{-\infty}^{\infty} \varphi_R^2(t) e^{-ixt} dt = \int_{-\infty}^{\infty} \psi_R(y) \overline{\psi_R(x+y)}\, dy,$$

where $\psi_R(y)$ is the Fourier transform of $\varphi_R(t)$.
And by $f_R(0) = 0$,

$$\int_{-\infty}^{\infty} \varphi_R^2(t)\, dt = \int_{-\infty}^{\infty} |\psi_R(y)|^2 dy = 1 .$$

Hence $f_R(t)$ is of the from (3.2). Thus the theorem is proved.

Theorem 5. (H Cramér)[1]. The necessary and sufficient condition for that a bounded function $f(t)$ with $f(0) = 1$ can be represented as (1.1), is that $g_\varepsilon(x) \geq 0$ for every $0 < \varepsilon < 1$, where

$$(3.5) \qquad g_\varepsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \mu(\varepsilon t) f(t)\, dt,$$

$\mu(t)$ being any function satisfying the following conditions;

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(1) H. Cramér, On the representation of a function by certain Fourier integrals, Trans. Amer. Math. Soc. 46 (1939); A. G. Dominguez, The representation of functions by the Fourier integrals, Duke Math. Journ., 6 (1940)

$$(3.6) \qquad \int_{-\infty}^{\infty} |\mu(t)|\, dt < \infty,$$

$$(3.7) \qquad \mu(t) = \int_{-\infty}^{\infty} e^{itx} m(x)\, dx, \quad \text{for some } m(x) \geqq 0,$$

and

$$(3.8) \qquad \mu(0) = \int_{-\infty}^{\infty} m(x)\, dx = 1.$$

Theorem 6. (H. Cramér)[1] The necessary and sufficient condition for that a bounded function $f(t)$ can be represented as (1.2) is that

$$(3.9) \qquad \int_{-\infty}^{\infty} |g_\varepsilon(x)|\, dx < M, \quad \text{for every } 0 < \varepsilon < 1,$$

M being independent of ε, where $g_\varepsilon(x)$ is the function of the form (3.5) with conditions (3.6) and (3.7)

These theorems can also be proved by using Theorems 1 and Theorems 5. The arguments are similar as in the proofs of the same theorems in the former paper. And so we shall leave the proofs to the readers.

4. We shall prove one more theorem which runs:

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(1) H. Cramér, loc. cit., A. G. Domingues loc. cit..

Theorem 7. In order that a continuous and bounded function $f(t)$ can be written as (1.2), it is necessary and sufficient that for any function $\varphi(t)$ belonging to $L(-\infty,\infty)$ it holds that

$$(4.1)\qquad \left|\int_{-\infty}^{\infty}\varphi(t)f(t)\,dt\right| \leq M\cdot\sup_{-\infty<x<\infty}\left|\int_{-\infty}^{\infty}\varphi(t)e^{ixt}\,dt\right|$$

where M is a constant independent of $\varphi(t)$.

This is the analogous theorem to one due to S. Bochner[1], in which (4.1) is replaced by the condition that for every finite sequence of real numbers $x_1, x_2, \dots, x_m$ and for every finite sequence of complex numbers $c_1, c_2, \dots, c_m$,

$$(4.2)\qquad \left|\sum_{\nu=1}^{m} c_\nu f(x_\nu)\right| \leq M\cdot\sup_{-\infty<x<\infty}\left|\sum_{\nu=1}^{m} c_\nu e^{ixx_\nu}\right|,$$

where M is a constant independent of the sets $\{x_\nu\}$, $\{c_\nu\}$.

Theorem 7 can be proved by using Cramér's theorem 6. But we can also deduce it from Theorem 2. The necessity of (4.1) is easily verified by putting (1.2) into it. So we prove the sufficiency.

(1) S. Bochner, A theorem on Fourier integral, Bull. Amer. Math. Soc., 1934, p. 271–276.

Since the function $f(t, A) = f(t)(1 - \frac{|t|}{A})$, $(|t| \leqq A)$ and $= 0$ $(|t| > A)$ ~~converges to~~ $f(t)$ as $A \to \infty$, we have only to prove that $f(t, A)$ can be represented as (1.2). Taking $R \geqq A$, we have.

$$\int_{-N}^{N} dx \left| \int_{-A}^{A} f(t, A) e^{-ixt} dt \right| = \int_{-N}^{N} \left| \int_{-A}^{A} f(t, A) e^{-ixt} dt \right| dx$$

which is written as, by putting [illegible]

$\psi(x) = \mathrm{sgn} \int_{-A}^{A} f(t, A) e^{-ixt} dt$ (1)

$$= \int_{-N}^{N} \psi(x) dx \int_{-A}^{A} f(t, A) e^{-ixt} dt$$

$$= \int_{-N}^{N} \psi(x) dx \int_{-A}^{A} (1 - \frac{|t|}{A}) e^{-ixt} f(t) dt$$

$$= \int_{-A}^{A} f(t)(1 - \frac{|t|}{A}) dt \int_{-N}^{N} \psi(x) e^{-ixt} dx.$$

By (4.1) with

$$\phi(t) = (1 - \frac{|t|}{A}) \int_{-N}^{N} \psi(x) e^{-ixt} dx, \quad \text{for } |t| \leqq A,$$

$$= 0, \quad \text{for } |t| > A.$$

ϕ tp Fomie [illegible]

we have

$$\int_{-N}^{N} dx \left| \int_{-R}^{R} f(t, A) e^{-ixt} dt \right|$$

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(1) $\mathrm{sgn}\, g(x) = \frac{\overline{g(x)}}{|g(x)|}$, if $g(x) \neq 0$ and $\mathrm{sgn}\, g(x) = 0$, if $g(x) = 0$,

$$\leq M \cdot \sup_{-\infty<y<\infty} \left| \int_{-A}^{A} e^{ity}\left(1-\frac{|t|}{A}\right)dt \int_{-N}^{N} \psi(x) e^{-ixt} dx \right|$$

$$= M \cdot \sup_{-\infty<y<\infty} \left| \int_{-N}^{N} \psi(x) dx \int_{-A}^{A} \left(1-\frac{|t|}{A}\right) e^{-i(x-y)t} dt \right|$$

$$= M \cdot \sup_{-\infty<y<\infty} \int_{-N}^{N} \left| \int_{-A}^{A} \left(1-\frac{|t|}{A}\right) e^{-i(x-y)t} dt \right| dx$$

$$= M \cdot \sup_{-\infty<y<\infty} \frac{1}{\pi} \int_{-N}^{N} \frac{\sin^2(A(x-y)/2)}{A(x-y)^2/2} dx$$

$$\leq M.$$

Hence we get

$$\int_{-\infty}^{\infty} dx \left| \int_{-R}^{R} f(t, A) e^{-ixt} dt \right| \leq M,$$

which proves the theorem.