Mean concentration function ト Quasi mean concentration function II

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§ 1.2. Uniform diminution theorem.

random variables ノ system

$$(1.2.1)\quad \begin{matrix} X_{11}, & X_{12}, & \cdots, & X_{1m_1} \\ X_{21}, & X_{22}, & \cdots, & X_{2m_2} \\ \cdots & \cdots & \cdots & \cdots \\ X_{n1}, & X_{n2}, & \cdots, & X_{nm_n} \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

ガ与ヘラレ此レヲ簡単ノタメ $\|X_{nm}\|$ デ書ク事ニスル。X_{nm} ノ分布函数ヲ $F_{nm}(x)$ トスル、今後 $\|X_{nm}\|$ ノ同行 $X_{n1}, \cdots, X_{nm_n}$ ハ互ニ独立トスルガ異列ノ random variables ハ必ズシモ独立トハ假定シナイ。サテ

定理 1.2.1. (Uniform diminution theorem)

二個ノ real numbers α ト β $(0<\alpha<1, 0<\beta\leq 1)$ ガ与ヘラレタトスル、サテ次ノ性質ヲモツ二個ノ正数 K ト N (両者共ニ α ト β トニ depend スル) ヲ定メル事ガ出来ル。即チ若シ $n \geq N$ ニシテ

$\{F_{n1}(x), F_{n2}(x), \cdots, F_{nm_n}(x)\}$ ガ

$$(1.2.2)\qquad \frac{1}{m_n}\sum_{k=1}^{m_n} \psi_{F_{nk}}(l_0) \leq \alpha$$

ヲ満足スル分布函数ノ集合ナラバ

$$\psi_{F_{n1} * \cdots * F_{nm_n}}(\sqrt{m_n}\, K l_0) \leq \beta$$

ガ成立スル。t_0 ハ任意ノ正数トスル。

$f_{nm}(t)$ ヲ $F_{nm}(x)$ ノ特性函数トシ $\widetilde{F}_{nm}(x)$ ヲ $F_{nm}(x)$ ノ対称化サレタ分布函数トスルナラバ

$$\left|\prod_{m=1}^{m_n} f_{nm}(t)\right|^2 \le \left(\frac{1}{m_n}\sum_{m=1}^{m_n} |f_{nm}(t)|^2\right)^{m_n}$$
$$\le \left(\int_{-\infty}^{\infty} e^{itx}\, d\left(\frac{1}{m_n}\sum_{m=1}^{m_n} \widetilde{F}_{nm}(x)\right)\right)^{m_n}$$

ガ成立スルカラ

$$\psi_{F_{n1}*F_{n2}*\cdots*F_{nm_n}}(l) \equiv l\int_0^{\infty} e^{-lt}\left|\prod_{m=1}^{m_n} f_{nm}(t)\right|^2 dt \le$$
$$\le l\int_0^{\infty} e^{-lt}(g_n(t))^{m_n}\, dt = \int_0^{\infty} e^{-t}\left(g_n\left(\frac{t}{l}\right)\right)^{m_n} dt \equiv \varphi_n(l),$$

此處ニ

$$g_n(t) \equiv \int_{-\infty}^{\infty} e^{itx}\, d\left(\frac{1}{m_n}\sum_{k=1}^{m_r} \widetilde{F}_{rk}(x)\right)$$

(1, 2, 5) $$\equiv \int_{-\infty}^{\infty} e^{itx}\, dG_n(x)$$

ハ対称ナル分布函数 $G_n(x)$ ノ特性函数デアル。ル。此ノ事ヨリ (1, 2, 3) ヲ示スニハ

$$\varphi_n(\sqrt{m_n}\, K t_0) \ge \beta,\quad n \ge N$$

ヲ証明スレバ充分デアル。

<u>補助定理 1.2.1</u>　m ヲ与ヘラレタ正ノ整数トシ順次ニ

(1, 2, 6) $$0 < \delta < \frac{1-\alpha}{2}$$

(1, 2, 7) $$\varepsilon \equiv \frac{-\delta\beta}{\sqrt{3\pi}\,\log\{\beta(1-\delta)\}}$$

且ツ

(1, 2, 8) $$h \equiv \left(\frac{3}{\varepsilon}\right)^2$$

トオク, 若シ $m_n \geq mh$ ヲ満足シ

(1, 2, 9) $\quad \delta^\circ = G_n^{m*}(3a l_0) - G_n^{m*}(a l_0)$ [1]

ナル様ナ実数 $a(\geq \sqrt{m_n}\,\varepsilon)$ ガ存在スルナラバ

$$\varphi_n(\sqrt{m_n}\,\tau l_0) < \beta, \qquad m_n \geq mh$$

ガ成立スル

此處ニ

(1, 2, 10) $$\tau = \frac{2\delta\beta}{\sqrt{3}\,\pi}$$

ハ α ト β ノミニ depend スル常数ヲ示ス。

<u>証明.</u> (1, 2, 5) カラ

$$(g_n(t))^m = \int_{-\infty}^{\infty} e^{itx} d\,G_n^{m*}(x) = 1 + \int_{|x| \leq \sqrt{h}\,a\varepsilon l_0} (e^{itx} - 1)\,dG_n^{m*}(x)$$

$$+ \int_{|x| > \sqrt{h}\,a\varepsilon l_0} (e^{itx} - 1)\,d\,G_n^{m*}(x)$$

$G_n^{m*}(x)$ ハ対称デアルカラ最後ノ項ハ ≤ 0 デアル。故ニ

$$(g_n(t))^m \leq 1 - t^2 \int_0^{\sqrt{h}\,a\varepsilon l_0} x^2 d\,G_n^{m*}(x) + \frac{t^3}{3}\int_0^{\sqrt{h}\,a\varepsilon l_0} x^3 d\,G_n^{m*}(x)$$

サテ

(1, 2, 11) $\quad T \equiv -\log\{\beta(1 - \sqrt{\delta})\}$

トオクト $0 \leq t \leq T$ ナル如何ナル t ニ対シテモ

(1, 2, 12) $$\left(g_n\left(\frac{t}{\sqrt{h}\,a\tau l_0}\right)\right)^m \leq 1 - \frac{t^2}{h a^2 \tau^2 l_0^2}\int_0^{\sqrt{h}\,a\varepsilon l_0} x^2 dG_n^{m*}(x)$$

$$\leq 1 - \frac{2t^2}{3 h a^2 \tau^2 l_0^2}\int_0^{\sqrt{h}\,a\varepsilon l_0} x^2 d\,G_n^{m*}(x)$$

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1) G^{m*} ハ m 個ノ equal components G ノ convolution $G * G * \cdots \cdots * G$ ヲ示ス.

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(1.2.9) ト (1.2.8) ヨリ

$$\delta \leq G_n^{m*}(3al_0) - G_n^{m*}(al_0)$$
$$\leq \frac{1}{a^2l_0^2}\int_{al_0}^{3al_0} x^2 dG_n^{m*}(x) \leq \frac{1}{a^2l_0^2}\int_0^{\sqrt{h}a\varepsilon l_0} x^2 dG_n^{m*}(x)$$

故ニ (1, 2, 12) カラ

$$\left(g_n\left(\frac{t}{\sqrt{h}a\tau l_0}\right)\right)^m \leq \frac{2t^2\delta}{3h\tau^2} \leq e^{-\frac{2t^2\delta}{3h\tau^2}} \qquad 0 \leq t \leq T.$$

今 $m_n = mh$ ナルコトヲ使ッテ

$$\left(g_n\left(\frac{t}{\sqrt{h}a\tau l_0}\right)\right)^{m_n} \leq \left(g_n\left(\frac{t}{\sqrt{h}a\tau l_0}\right)\right)^{mh}$$
$$\leq e^{-\frac{2t^2\delta}{3\tau^2}}, \qquad 0 \leq t \leq T.$$

従ッテ

$$\varphi_n(\sqrt{h}a\tau l_0) = \int_0^\infty e^{-t}\left(g_n\left(\frac{t}{\sqrt{h}a\tau l_0}\right)\right)^n dt \leq$$
$$\leq \int_0^T e^{-t}\left(g_n\left(\frac{t}{\sqrt{h}a\tau l_0}\right)\right)^n dt + \int_T^\infty e^{-t}dt \leq$$
$$\leq \int_0^T e^{-t}e^{-\frac{2t^2\delta}{3\tau^2}} dt + e^{-T} \leq$$
$$\leq \frac{\sqrt{3\pi}\,\tau}{2\sqrt{2}\sqrt{\delta}} + e^{\log\{\beta(1-\sqrt{\delta})\}}$$

(1, 2, 6), δ 及 τ ノ定義ト (1.2, 1) ヨリ

$$\varphi_n(\sqrt{h}a\tau l_0) \leq \frac{\beta\sqrt{\delta}}{\sqrt{2}} + \beta - \beta\sqrt{\delta} < \beta$$

$\sqrt{m_n} < \frac{3}{\varepsilon}\sqrt{m_n}\,\varepsilon < \sqrt{h}\,a$ ナル故

$$\varphi_n(\sqrt{m_n}\,\tau l_0) \leq \varphi_n(\sqrt{h}\,\tau l_0) \leq \beta,$$
$$m_n \geq mh$$

ガ出来ル。

定理 1.2.1ノ証明　今後 n ハ任意ニ固定サレタ整数トシ

(1.2.13) $\quad m_n > N \equiv N(\alpha, \beta) = h\left\{1 + \dfrac{\log\left(1 - \frac{(2-\beta)(1-\alpha-2\delta)}{2(1-\alpha-5\delta)}\right)}{\log(\alpha+2\delta)}\right\}$

トスル。

又 $\delta, \varepsilon, \tau$ 及ビ T ハ既ニ定メラレタモノト同ジモノデアル。先ヅ次ノ二個ノ場合ガ考ヘラレル。

(1.2.14₁) $\quad \bar{G}_n(\sqrt{m_n}\,\varepsilon l_0) \equiv 1 - G_n(\sqrt{m_n}\,\varepsilon l_0) \leq \dfrac{1-\alpha-\delta}{2}$

(1.2.14₂) $\quad \bar{G}_n(\sqrt{m_n}\,\varepsilon l_0) > \dfrac{1-\alpha-\delta}{2}$

1°. 先ヅ (1.2.14₁) ヲ假定スル。(1.2.2) ヨリ

$$1-\alpha \leq 2\int_0^\infty \frac{x^2}{l_0^2+x^2}\, d\left(\frac{1}{m_n}\sum_{m=1}^{m_n} \tilde{F}_{nm}(x)\right) =$$

$$= 2\int_0^\infty \frac{x^2}{l_0^2+x^2}\, dG_n(x) \leq \frac{2}{l_0^2}\int_0^{\sqrt{m_n}\,\varepsilon l_0} x^2\, dG_n(x) + 2\bar{G}_n(\sqrt{m_n}\,\varepsilon l_0)$$

$$\leq \frac{2}{l_0^2}\int_0^{\sqrt{m_n}\,\varepsilon l_0} x^2\, dG_n(x) + 1 - \alpha - \delta.$$

従ッテ

$$\delta \leq \frac{2}{l_0^2}\int_0^{\sqrt{m_n}\,\varepsilon l_0} x^2\, dG_n(x)$$

$$g_n(t) \equiv \int_{-\infty}^{\infty} e^{itx}\, dG_n(x) = 1 + \int_{|x| \leq \sqrt{m_n}\,\varepsilon l_0} (e^{itx}-1)\, dG_n(x)$$

$$+ \int_{|x| > \sqrt{m_n}\,\varepsilon l_0} (e^{itx}-1)\, dG_n(x)$$

補助定理 1.2.1ノ証明ト同ジ論法デ

$$g_n\left(\frac{t}{\sqrt{m_n}\,\tau\, l_0}\right) \leq 1 - \frac{2t^2}{3m_n\,\tau^2 l_0^2}\int_0^{\sqrt{m_n}\,\varepsilon l_0} x^2\, dG_n(x) \leq 1 - \frac{t^2\delta}{3m_n\,\tau^2}$$

$$\leq e^{-\frac{t^2\delta}{3m_n\tau^2}}, \quad 0 \leq t \leq T$$

従ッテ

$$\varphi_n(\sqrt{m_n}\tau l_0)=\int_0^\infty e^{-t}\left(g_n\left(\frac{t}{\sqrt{m_n}\tau l_0}\right)\right)^{m_n}dt \leq$$
$$\leq \int_0^T e^{-t}\left(g_n\left(\frac{t}{\sqrt{m_n}\tau l_0}\right)\right)^{m_n}dt+\int_T^\infty e^{-t}dt$$
$$\leq \int_0^T e^{-t}e^{-\frac{t^2\delta}{3\tau^2}}dt+\int_T^\infty e^{-t}dt \leq \frac{\sqrt{3\pi}}{2}\frac{\tau}{\sqrt{\delta}}+e^{\log\{\beta(1-\sqrt{\delta})\}}$$
$$\leq \beta\sqrt{\delta}+\beta(1-\sqrt{\delta})=\beta.$$

2° (1.2.14₂) ヲ假定シヨウ。　如何ナル

$a(\geq\sqrt{m_n}\varepsilon)$ ニ対シテモ

$$(1.2.15)\quad G_m^{m*}(3al_0)-G_n^{m*}(al_0)<\delta$$
$$m=1,2,\cdots,\left[\frac{N}{h}\right]$$

ト假定シテヨイ。此處ニ $\left[\frac{N}{h}\right]$ ハ $\frac{N}{h}$ ノ整数部分ヲ示ス。何トナラバ (1.2.9) ガ成立スル様ナ $a(\geq\sqrt{m_n}\varepsilon)$ ト $m\left(1\leq m\leq\left[\frac{N}{h}\right]\right)$ ガ存在スルナラバ補助定理1.2.1ヨリ $m_n\geq N\geq mh$ ヲ使ッテ $\varphi_n(\sqrt{m_n}\tau l_0)<\beta$ ガ出ルカラ。

サテ (1.2.15) ニ於テ $m=1$ トオイテ

$y=2al_0>al_0\geq\sqrt{m_n}\varepsilon l_0$ ニ対シ

$$\begin{aligned}&\bar{G}_n(y-\sqrt{m_n}\varepsilon l_0)-\bar{G}_n(y+\sqrt{m_n}\varepsilon l_0)\\&=G_n(y+\sqrt{m_n}\varepsilon l_0)-G_n(y-\sqrt{m_n}\varepsilon l_0)\\&\leq G_n(3al_0)-G_n(al_0)<\delta\end{aligned}\qquad(1.2.16)$$

従ッテ

$$\bar{G}_n^{2*}(\sqrt{m_n}\varepsilon l_0)\equiv 1-G_n^{2*}(\sqrt{m_n}\varepsilon l_0)$$
$$=\int_{-\infty}^{\infty}\bar{G}_n(\sqrt{m_n}\varepsilon l_0-y)\,dG_n(y)$$
$$=\int_{-\infty}^{0}\bar{G}_n(\sqrt{m_n}\varepsilon l_0-y)\,dG_n(y)+\int_0^\infty\bar{G}_n(\sqrt{m_n}\varepsilon l_0+y)\,dG_n(y)$$

$$(1.2.17) = \int_0^{2\sqrt{m_n}\varepsilon l_0} \bar{G}_n(\sqrt{m_n}\varepsilon l_0 - y)\, dG_n(y)$$
$$+ \int_{2\sqrt{m_n}\varepsilon l_0}^{\infty} \bar{G}_n(\sqrt{m_n}\varepsilon l_0 - y)\, dG_n(y)$$
$$+ \int_0^{2\sqrt{m_n}\varepsilon l_0} \bar{G}_n(\sqrt{m_n}\varepsilon l_0 + y)\, dG_n(y)$$
$$+ \int_{2\sqrt{m_n}\varepsilon l_0}^{\infty} (\bar{G}_n(y - \sqrt{m_n}\varepsilon l_0) - \delta)\, dG_n(y)$$

従ツテ

$\bar{G}_n(\sqrt{m_n}\varepsilon l_0 - y) + \bar{G}_n(y - \sqrt{m_n}\varepsilon l_0) = G_n(y - \sqrt{m_n}\varepsilon l_0) +$
$+ \bar{G}_n(y - \sqrt{m_n}\varepsilon l_0) = 1$ ヲ利用シテ

$$\bar{G}_n^{2*}(\sqrt{m_n}\varepsilon l_0) \geq \bar{G}_n(\sqrt{m_n}\varepsilon l_0)\left\{G_n(2\sqrt{m_n}\varepsilon l_0) - \frac{1}{2}\right\}$$
$$+ \bar{G}_n(3\sqrt{m_n}\varepsilon l_0)\left\{G_n(2\sqrt{m_n}\varepsilon l_0) - \frac{1}{2}\right\} +$$
$$+ (1-\delta)\bar{G}_n(2\sqrt{m_n}\varepsilon l_0) \geq \left\{\bar{G}_n(\sqrt{m_n}\varepsilon l_0) + \right.$$
$$\left.\bar{G}_n(3\sqrt{m_n}\varepsilon l_0)\right\}\left\{G_n(2\sqrt{m_n}\varepsilon l_0) - \frac{1}{2}\right\} +$$
$$+ (1-\delta)\bar{G}_n(2\sqrt{m_n}\varepsilon l_0)$$

(1.2.15) ニ於イテ $a = \sqrt{m_n}\varepsilon$ ト $m = 1$ トオイテ (1.2.14₂) ヲ使ツテ

$$\frac{1-\alpha-3\delta}{2} = \frac{1-\alpha-\delta}{2} - \delta < \bar{G}_n(\sqrt{m_n}\varepsilon l_0) -$$
$$-\delta < \bar{G}_n(3\sqrt{m_n}\varepsilon l_0)$$

故ニ

$$\bar{G}_n^{2*}(\sqrt{m_n}\varepsilon l_0) \geq (1-\alpha-3\delta)\left\{G_n(2\sqrt{m_n}\varepsilon l_0) - \frac{1}{2}\right\}$$
$$+ \left\{(1-\alpha-3\delta) + (\alpha+2\delta)\right\}\bar{G}_n(2\sqrt{n}\varepsilon l_0)$$
$$\geq \frac{1-\alpha-3\delta}{2} + (\alpha+2\delta)\left(\frac{1-\alpha-\delta}{2} - \delta\right) = \frac{1-\alpha-3\delta}{2}(1+\eta)$$
$$\eta = \alpha + 2\delta.$$

故ニ (1, 2, 14) ヨリ (1, 2, 15) ニ於テ $a=\sqrt{m_n}\,\varepsilon$, $m=2$ トオキ

(1, 2, 18) $\frac{1-\alpha-3\delta}{2}(1-\eta)-\delta \leq \bar{G}_n^{2*}(\sqrt{m_n}\,\varepsilon\, l_0)-\delta < \bar{G}^{2*}(3\sqrt{m_n}\,\varepsilon\, l_0)$

(1, 2, 18) ガ得ラレタノト同様ナ方法デ

$$\frac{1-\alpha-3\delta}{2}(1+\eta+\eta^2)-\delta\,(1+\eta) < \bar{G}_n^{3*}(\sqrt{m_n}\,\varepsilon\, l_0)$$

此ノ論法ヲクリ返シテ最後ニ

$$\frac{1-\alpha-3\delta}{2}\left(\sum_{k=0}^{[\frac{N}{n}]-1}\eta^k\right)-\delta\left(\sum_{k=0}^{[\frac{N}{n}]-2}\eta^k\right) \leq$$

$$\leq \bar{G}_n^{[\frac{N}{n}]*}(\sqrt{m_n}\,\varepsilon\, l_0);$$

$\eta=\alpha+2\delta<\alpha+(1-\alpha)=1$ ナル故

$$\bar{G}_n^{[\frac{N}{n}]*}(\sqrt{m_n}\,\varepsilon\, l_0) \geq \frac{1-\alpha-5\delta}{2}\left(\sum_{k=0}^{[\frac{N}{n}]-1}\eta^k\right)=$$

$$=\frac{1-\alpha-5\delta}{2}\,\frac{1-\eta^{[\frac{N}{n}]}}{1-\eta} \geq \frac{1-\alpha-5\delta}{2}\,\frac{1-\eta^{(\frac{N}{n}-1)}}{1-\eta}$$

(1, 2, 13) ヨリ

(1, 2, 19) $\bar{G}_n^{[\frac{N}{n}]*}(\sqrt{m_n}\,\varepsilon\, l_0) \geq \frac{1-\alpha-5\delta}{2}\,\frac{1-\eta^{(\frac{N}{n}-1)}}{1-\eta}$

$$=\frac{2-\beta}{4}$$

(1, 2, 20) $\lambda=\sqrt{\frac{\varepsilon^2\beta}{2(1-\beta)}}$

トオク、(1, 2, 19) ト mean concentration function ハ分布函数ノ covolution ニヨリ減少スル事実ヨリ

$$1-\varphi_n(\sqrt{m_n}\,\lambda\, l_0)=2\int_0^\infty \frac{x^2}{(\sqrt{m_n}\,\lambda\, l_0)^2+x^2}\,dG_n^{m_n*}(x) \geq$$

$$\geq 2\int_0^\infty \frac{x^2}{(\sqrt{m_n}\lambda l_0)^2+x^2}\, dG_n^{[\frac{N}{2}]*} \geq$$

$$\geq 2\int_{\sqrt{m_n}\varepsilon l_0}^\infty \frac{x^2}{(\sqrt{m_n}\lambda l_0)^2+x^2}\, dG_n^{[\frac{N}{2}]*}(x) \geq$$

$$\geq \frac{2\varepsilon^2}{\varepsilon^2+\lambda^2}\, \bar{G}_n^{[\frac{N}{2}]*}(\sqrt{m_n}\varepsilon l_0) \geq \frac{\varepsilon^2(2-\beta)}{2(\varepsilon^2+\lambda^2)}$$

(1, 2, 20) ヨリ

$$\frac{(2-\beta)\,\varepsilon^2}{2(\varepsilon^2+\lambda^2)} = \frac{(2-\beta)\,\varepsilon^2}{2\left(\varepsilon^2+\frac{\varepsilon^2\beta}{2(1-\beta)}\right)} = 1-\beta.$$

從ツテ

$$\varphi_n(\sqrt{m_n}\lambda l_0) \leq \beta, \quad n \geq N$$

λノ定義ヨリλハαトβノミニdependスル定数ナル事ガワカル。依ツテ$K = \min(\lambda, \tau)$トオキ(1, 2, 4)ヲ満足スルnニ対シ

$$\psi_{F_{n1}*\cdots*F_{nm_n}}(\sqrt{m_n}K l_0) \leq \beta$$

ヲ結論スル事ガ出来ル。

<u>系 1, 2, 1.</u> 互ニ独立ナrandom variablesノ系列$\{x_k \mid k=1,2,\cdots\}$ト二個ノ実数αトβ ($0<\alpha<1$, $0<\beta\leq 1$) ガ与ヘラレタトスル。スルト次ニ述ベル様ナαトβノミニdependスルKトNヲ定メル事ガ出来ル、即チ若シ$n \geq N$ニシテ且$\{F_1(x), \cdots, F_n(x)\}$ガ

$$\psi_{F_k}(l_0) \leq \alpha, \quad k=1,2,\cdots,n$$

ヲ満足スル$\{x_1,\cdots,x_n\}$ノ分布函数ノ畳

スルナラバ

$$\psi_{F_1 * \cdots F_n}(\sqrt{n}\, K\, l_0) \le \beta$$

ガ成立スル。此處ニ l_0 ハ任意ノ固定サレタ正数トスル。

此ノ系ニ於テ ψ_F ヲ Q_F ニ置換ヘレバ Lévy-Doeblin ノ定理ヲ得ル。然シ吾々ノ場合デハ Lévy-Doeblin ノ場合デ与ヘラレナカツタ N ト K ノ具体的ナ値ヲ定メル事ガ出来タ（第一章 3）。（終）