Dirichlet 分布について

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On the Dirichlet Distribution

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In this paper, we consider the properties of generalized Dirichlet distribution of matrix argument. The original density function of this was given by I. Olkin and H. Rubin. We think that it is useful to arrange their properties here for the point of statistical education.

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1. 目　的

適当な自由度を持つ $k+1$ コの独立な χ^2 変数を変換することにより，Dirichlet 分布を得ることが出来ることはよく知られている Wilks [1]. 本論文は $k+1$ コの独立な Wishart 行列より一般化された Dirichlet 分布を取りあつかうことを目的としている．しかし，この問題の本質的な所は I. Olkin and H. Rubin [2] の［定理 3］に与えられている．故に全てここで述べる結果は I. Olkin and H. Rubin によるものから出てくるのである．これらを Wilks [1] の第7章7節に有る順に従って述べることとする．

2. 記　号

(1)　正値符号に関する Loewner の意味による順序づけとして，S が正値定符号の時，$S>0$ で表し，$S-T>0$ のとき，$S>T$ で示す．

(2)　多変数ガンマ函数として，

$$\Gamma_p(a)=\pi^{1/4p(p-1)}\prod_{i=1}^{p}\Gamma\left[a-\frac{1}{2}(i-1)\right]$$

を用いる．

(3)　Zonal 多項式を $C_\kappa(S)$ で表す．$C_\kappa(S)$ は，対称行列の固有根の k 次の同次多項式で，k の分割 κ に対応するものである．James [3], Constantine [4].

(4)　対称行列は全て $p\times p$ 行列とする．

(5)　$W(\Sigma,n)$ は自由度 n の共分散 Σ を持つ p 次の Wishart 分布を示し，$S\in W(\Sigma,n)$ は対称行列 S が上述の Wishart 分布をすることを意味する．

3. Dirichlet 分布

［定理 1］ (I. Olkin and H. Rubin)

$S_i\in W(\Sigma,n_i)$, $(i=1,2,\cdots\cdots,k+1)$ は互に独立とするとき，

$$V_j=\Big(\sum_{i=1}^{k+1}S_i\Big)^{-1/2}S_j\Big(\sum_{i=1}^{k+1}S_i\Big)^{-1/2},\quad (j=1,2,\cdots\cdots,k),$$

の同時密度函数は

$$p(V_1, \cdots, V_k) = \frac{\Gamma_p\left(\frac{n}{2}\right)}{\prod_{i=1}^{k+1} \Gamma_p\left(\frac{n_i}{2}\right)} \prod_{j=1}^{k} |V_j|^{1/2(n_j-p-1)} \left|I - \sum_{j=1}^{k} V_j\right|^{1/2(n_{k+1}-p-1)},$$

で与えられる．但し，$n = \sum_{i=1}^{k+1} n_i$．

証明は省略する．

Dirichlet 分布を $D(n_1, \cdots, n_k; n_{k+1})$ で表す．

(補題 1)

$(V_1, \cdots, V_k) \in D(n_1, \cdots, n_k; n_{k+1})$ とするとき，

$$E\left(\prod_{j=1}^{k} |V_j|^{r_j}\right) = \frac{\Gamma_p\left(\frac{n}{2}\right) \prod_{j=1}^{k} \Gamma_p\left(\frac{n_j}{2} + r_j\right)}{\Gamma_p\left(\frac{n}{2} + \sum_{j=1}^{k} r_j\right) \prod_{j=1}^{k} \Gamma_p\left(\frac{n_j}{2}\right)},$$

で与えられる．

(証明)

$$E\left(\prod_{j=1}^{k} |V_j|^{r_j}\right) = \frac{\Gamma_p\left(\frac{n}{2}\right)}{\prod_{j=1}^{k+1} \Gamma_p\left(\frac{n_j}{2}\right)} \int \cdots \int \prod_{j=1}^{k} |V_j|^{1/2(n_j+2r_j-p-1)} \left|I - \sum_{j=1}^{k} V_j\right|^{1/2(n_{k+1}-p-1)} dV_1 \cdots dV_k$$

$$= \frac{\Gamma_p\left(\frac{n}{2}\right)}{\prod_{j=1}^{k+1} \Gamma_p\left(\frac{n_j}{2}\right)} \frac{\prod_{j=1}^{k} \Gamma_p\left(\frac{n_j + 2r_j}{2}\right) \Gamma_p\left(\frac{n_{k+1}}{2}\right)}{\Gamma_p\left(\frac{n}{2} + \sum_{j=1}^{k} r_j\right)}.$$

(補題 2)

$$E(|V_j|) = \frac{\Gamma_p\left(\frac{n}{2}\right) \Gamma_p\left(\frac{n_j}{2} + 1\right)}{\Gamma_p\left(\frac{n}{2} + 1\right) \Gamma_p\left(\frac{n_j}{2}\right)} \quad (j = 1, 2, \cdots, k).$$

$$Var(|V_j|) = \frac{\Gamma_p\left(\frac{n}{2}\right)}{\Gamma_p\left(\frac{n_j}{2}\right)} \left\{ \frac{\Gamma_p\left(\frac{n_j}{2} + 2\right)}{\Gamma_p\left(\frac{n}{2} + 2\right)} - \frac{\Gamma_p\left(\frac{n}{2}\right) \Gamma_p^2\left(\frac{n_j}{2} + 1\right)}{\Gamma_p^2\left(\frac{n}{2} + 1\right) \Gamma_p\left(\frac{n_j}{2}\right)} \right\}, \quad (j = 1, 2, \cdots, k)$$

$$Cov(|V_i|, |V_j|) = \frac{\Gamma_p\left(\frac{n}{2}\right) \Gamma_p\left(\frac{n_1}{2} + 1\right) \Gamma_p\left(\frac{n_2}{2} + 1\right)}{\Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right)} \left\{ \frac{1}{\Gamma_p\left(\frac{n}{2} + 2\right)} - \frac{\Gamma_p\left(\frac{n}{2}\right)}{\Gamma_p^2\left(\frac{n}{2} + 1\right)} \right\}.$$

補題 1 と，$0 < |V_j| < 1$ より，$|V_j|$ は独立な Beta 変数の積として表せる．即ち，

$$E(|V_j|^{r_j}) = \frac{\Gamma_p\left(\frac{n}{2}\right) \Gamma_p\left(\frac{n_j}{2} + r_j\right)}{\Gamma_p\left(\frac{n}{2} + r_j\right) \Gamma_p\left(\frac{n_j}{2}\right)} = \prod_{i=1}^{p} \frac{\Gamma\left(\frac{n-i+1}{2}\right) \Gamma\left(\frac{n_j - i + 1}{2} + r_j\right)}{\Gamma\left(\frac{n-i+1}{2} + r_j\right) \Gamma\left(\frac{n_j - i + 1}{2}\right)}$$

$$= \prod_{i=1}^{p} E\,(X_{ji}{}^{r_j})\,.$$

ここで，$X_{ji} \in \mathrm{Beta}\left(x_{ji}\,;\,\frac{n_j - i + 1}{2},\ \frac{n-n_j}{2}\right)$ $(i=1, 2, \cdots, p)$ は独立である．故に，

$$|V_j| = \prod_{i=1}^{p} X_{ji} \quad (j=1, 2, \cdots, k)$$

と表せる．

［定理 2］

$(V_1, \cdots, V_k) \in D\,(n_1, \cdots, n_k\,;\,n_{k+1})$ とするとき，$(V_1, \cdots V_s)$ $(s<k)$ の周辺密度函数は

$$p_1(V_1, \cdots, V_s) = \frac{\Gamma_p\left(\frac{n}{2}\right)}{\prod_{i=1}^{s}\Gamma_p\left(\frac{n_i}{2}\right)\Gamma_p\left(\frac{1}{2}\sum_{j=s+1}^{k+1} n_j\right)} \prod_{j=1}^{s} |V_j|^{1/2(n_j-p-1)} \left|I - \sum_{j=1}^{s} V_j\right|^{1/2\left(\sum_{j=s+1}^{k+1} n_j - p - 1\right)}$$

である．即ち，$(V_1, \cdots, V_s) \in D\,(n_1, \cdots, n_s\,;\,n_{s+1} + \cdots + n_{k+1})$

（証明）

$$\prod_{j=1}^{k} |V_j|^{1/2(n_j-p-1)} \left|I - \sum_{j=1}^{k} V_j\right|^{1/2(n_{k+1}-p-1)}$$

$$= \prod_{j=1}^{s} |V_j|^{1/2(n_j-p-1)} \left|I - \sum_{j=1}^{s} V_j - \sum_{s+1}^{k} V_j\right|^{1/2(n_{k+1}-p-1)} \prod_{j=s+1}^{k} |V_j|^{1/2(n_j-p-1)}$$

に於て，$U = I - \sum_{j=1}^{s} V_j$ とおく．

$V_j = U^{1/2}\,W_j\,U^{1/2}$ $(j=s+1,\ s+2, \cdots, k)$ とおくと，Jacobian は，

$$J\,(V_{s+1}, \cdots, V_k \to W_{s+1}, \cdots, W_k) = |U|^{1/2(p+1)(k-s)}$$

である．故に，

$$\prod_{s+1}^{k} |V_j|^{1/2(n_j-p-1)} \left|I - \sum_{j=1}^{s} V_j - \sum_{s+1}^{k} V_j\right|^{1/2(n_{k+1}-p-1)}$$

$$= |U|^{1/2\left(\sum_{s+1}^{k}(n_j-p-1)\right)} |U|^{1/2(n_{k+1}-p-1)} \left|I - \sum_{s+1}^{k} W_j\right|^{1/2(n_{k+1}-p-1)} \prod_{s+1}^{k} |W_j|^{1/2(n_j-p-1)} \cdot |U|^{1/2(p+1)(k-s)}$$

$$= |U|^{1/2\left(\sum_{s+1}^{k+1} n_j - p - 1\right)} \prod_{s+1}^{k} |W_j|^{1/2(n_j-p-1)} \left|I - \sum_{s+1}^{k} W_j\right|^{1/2(n_{k+1}-p-1)}\,.$$

ここで $(W_{s+1}, \cdots, W_k)$ について積分すれば，求めるものが得られる．

［補題 3］

$$E\left(\prod_{j=1}^{s} |V_j|^{r_j}\right) = \frac{\Gamma_p\left(\frac{n}{2}\right)\prod_{j=1}^{s}\Gamma_p\left(\frac{n_j}{2}+r_j\right)}{\Gamma_p\left(\frac{n}{2}+\sum_{i=1}^{s} r_j\right)\prod_{j=1}^{s}\Gamma_p\left(\frac{n_j}{2}\right)}\,.$$

［定理 3］

$(V_1, \cdots, V_k) \in D\,(n_1, \cdots, n_k\,;\,n_{k+1})$ とするとき，$(V_1, \cdots, V_s)$ given の $(V_{s+1}, \cdots, V_k)$ の条件付分布は，

$$(V_{s+1}, \cdots, V_k \,|\, V_1, \cdots, V_s) \in D\,(n_{s+1}, \cdots, n_k\,;\,n_{k+1})$$

となる.

(証明) 条件付密度函数は,

$$p(V_{s+1},\cdots,V_k|V_1,\cdots V_s)=\frac{p(V_1,\cdots,V_k)}{p_1(V_1,\cdots,V_s)}$$

で与えられる.

係数に関しては,

$$\frac{\Gamma_p\left(\frac{n}{2}\right)}{\prod_{j=1}^{k+1}\Gamma_p\left(\frac{n_j}{2}\right)}\Bigg/\frac{\Gamma_p\left(\frac{n}{2}\right)}{\prod_{i=1}^{s}\Gamma_p\left(\frac{n_i}{2}\right)\Gamma_p\left(\frac{1}{2}\sum_{j=s+1}^{k+1}n_j\right)}=\frac{\Gamma_p\left(\frac{1}{2}\sum_{j=s+1}^{k+1}n_j\right)}{\prod_{j=s+1}^{k+1}\Gamma_p\left(\frac{n_j}{2}\right)}.$$

変数に関しては,

$$\frac{\prod_{j=1}^{k}|V_j|^{1/2(n_j-p-1)}\left|I-\sum_{j=1}^{k}V_j\right|^{1/2(n_{k+1}-p-1)}}{\prod_{j=1}^{s}|V_j|^{1/2(n_j-p-1)}\left|I-\sum_{j=1}^{s}V_j\right|^{1/2\left(\sum_{s+1}^{k+1}n_j-p-1\right)}}$$

$$=\prod_{j=s+1}^{k}|V_j|^{1/2(n_j-p-1)}\frac{\left|I-\sum_{j=1}^{k}V_j\right|^{1/2(n_{k+1}-p-1)}}{\left|I-\sum_{j=1}^{s}V_j\right|^{1/2\left(\sum_{s+1}^{k+1}n_j-p-1\right)}}.$$

ここで,

$$\frac{\left|I-\sum_{j=1}^{s}V_j-\sum_{i=s+1}^{k}V_i\right|^{1/2(n_{k+1}-p-1)}}{\left|I-\sum_{j=1}^{s}V_j\right|^{1/2\left(\sum_{j=s+1}^{k+1}n_j-p-1\right)}}$$

$$=\frac{\left|I-\sum_{i=s+1}^{k}\left(I-\sum_{j=1}^{s}V_j\right)^{-1/2}V_i\left(I-\sum_{j=1}^{s}V_j\right)^{-1/2}\right|^{1/2(n_{k+1}-p-1)}}{\left|I-\sum_{j=1}^{s}V_j\right|^{1/2\left(\sum_{s+1}^{k}n_j\right)}}.$$

故に,

$$\begin{cases}U=I-\sum_{j=1}^{s}V_j\,,\\ V_j=U^{1/2}W_jU^{1/2}\,,\quad (j=s+1,\cdots,k),\end{cases}$$

と置くと, Jacobian は,

$$J(V_{s+1},\cdots,V_k\to W_{s+1},\cdots,W_k)=|U|^{1/2(p+1)(k-s)}.$$

故に,

$$\prod_{j=s+1}^{k}|V_j|^{1/2(n_j-p-1)}\frac{\left|I-\sum_{i=s+1}^{k}\left(I-\sum_{j=1}^{s}V_j\right)^{-1/2}V_i\left(I-\sum_{j=s+1}^{k}V_j\right)^{-1/2}\right|^{1/2(n_{k+1}-p-1)}}{\left|I-\sum_{j=1}^{s}V_j\right|^{1/2\left(\sum_{j=s+1}^{k}n_j\right)}}$$

$$= \prod_{j=s+1}^{k} |W_j|^{1/2(n_j-p-1)} \left|I - \sum_{j=s+1}^{k} W_j\right|^{1/2(n_{k+1}-p-1)}.$$

故に，$(W_{s+1}, \cdots, W_k) \in D(n_{s+1}, \cdots, n_k; n_{k+1})$ となる．

[補題 4]

$V_{s+1}, \cdots, V_k | V_1, \cdots, V_s$ の Moment

$$E\left(\prod_{j=s+1}^{k} |V_j|^{r_j} | V_1, \cdots, V_s\right)$$

$$= \frac{\Gamma_p\left(\frac{1}{2}\sum_{j=s+1}^{k+1} n_j\right) \prod_{j=s+1}^{k} \Gamma_p\left(\frac{n_j}{2} + r_j\right)}{\Gamma_p\left(\frac{1}{2}\sum_{j=s+1}^{k+1} n_j + \sum_{j=s+1}^{k} r_j\right) \prod_{j=s+1}^{k} \Gamma_p\left(\frac{n_j}{2}\right)} \left|I - \sum_{j=1}^{s} V_j\right|^{\sum_{s+1}^{k} r_j}.$$

特に，

$$U_1 = V_1,\ U_2 = V_1 + V_2, \cdots, U_k = V_1 + \cdots + V_k,$$

と置くとき，$0 < U_1 < U_2 < \cdots < U_k < I$ をみたす．$(U_1, \cdots, U_k)$ を ordered Dirichlet 分布と言う．これを $(U_1, \cdots, U_k) \in \text{ord-}D(n_1, \cdots, n_k; n_{k+1})$ で表す．

[定理 4]

$(U_1, \cdots, U_k) \in \text{ord-}D(n_1, \cdots, n_k; n_{k+1})$ とするとき，$(U_1, \cdots, U_k)$ の同時密度函数は，

$$p(U_1, \cdots, U_k) = \frac{\Gamma_p\left(\frac{n}{2}\right)}{\prod_{i=1}^{k+1} \Gamma_p\left(\frac{n_i}{2}\right)} |U_1|^{1/2(n_1-p-1)} |U_2 - U_1|^{1/2(n_2-p-1)} \cdots$$

$$\cdots |U_k - U_{k-1}|^{1/2(n_k-p-1)} |I - U_k|^{1/2(n_{k+1}-p-1)}.$$

特に，U_k は多変数 Beta 分布 $Be\left(\frac{1}{2}\sum_{j=1}^{k} n_j,\ \frac{1}{2} n_{k+1}\right)$ となる．

(証明)

定理 1 に於て，$V_1 = U_1$, $V_2 = U_2 - U_1, \cdots, V_k = U_k - U_{k-1}$ と置けば，Jacobian は 1 であるから，求めるものとなる．

さて，

$$\int_0^{U_2} |U_1|^{1/2(n_1-p-1)} |U_2 - U_1|^{1/2(n_2-p-1)} dU_1 = \frac{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2}{2}\right)} |U_2|^{1/2(n_1+n_2-p-1)}$$

であるから，

順次 $U_2 < U_3 < \cdots < U_{k-1} < U_k$ まで積分することにより，

$$\frac{\Gamma_p\left(\frac{n}{2}\right)}{\Gamma_p\left(\frac{1}{2}\sum_{j=1}^{k} n_j\right)\Gamma_p\left(\frac{n_{k+1}}{2}\right)} |U_k|^{1/2\left(\sum_{j=1}^{k} n_j - p - 1\right)} |I - U_k|^{1/2(n_{k+1}-p-1)}.$$

故に，U_k は多変数 Beta 分布となる．

[定理 5]

$(V_1, \cdots, V_k) \in D(n_1, \cdots, n_k; n_{k+1})$ とする．

$$Z_1 = V_1 + V_2 + \cdots + V_{k_1},\quad Z_2 = V_{k_1+1} + \cdots + V_{k_1+k_2}, \cdots,$$

$$Z_s = V_{k_1+\cdots+k_{s-1}+1} + \cdots + V_{k_1+\cdots+k_s},\quad k_1 + \cdots + k_s \le k,$$

とし，

$$n^{(1)}=n_1+\cdots\cdots+n_{k_1},\quad n^{(2)}=n_{k_1+1}+\cdots\cdots+n_{k_1+k_2},\ \cdots\cdots,$$
$$n^{(s)}=n_{k_1+\cdots+k_{s-1}+1}+\cdots\cdots+n_{k_1+\cdots+k_s}$$

と置く. このとき,

$$(Z_1, Z_2, \cdots\cdots, Z_s)\in D(n^{(1)}, \cdots\cdots, n^{(s)};\ n^{(s+1)})$$

となる. $n^{(s+1)}=n_{k_1+\cdots+k_s+1}+\cdots\cdots+n_{k+1}$ である.

(証明)

$$\begin{cases} U_1=V_1, \\ \quad\vdots \\ U_{k_1}=V_1+\cdots\cdots+V_{k_1}, \end{cases}$$
$$\begin{cases} W_1=V_{k_1+1}, \\ \quad\vdots \\ W_{k_2}=V_{k_1+1}+\cdots\cdots+V_{k_1+k_2}, \end{cases}$$
$$\vdots$$
$$\begin{cases} X_1=V_{k_1+\cdots+k_{s-1}+1}, \\ \quad\vdots \\ X_{k_s}=V_{k_1+\cdots+k_{s-1}+1}+\cdots\cdots+V_{k_1+\cdots+k_s}, \end{cases}$$

とおけば,

$$J(V_1, \cdots\cdots, V_k\rightarrow U_1, \cdots\cdots, U_{k_1}, W_1, \cdots\cdots W_{k_2}, \cdots\cdots, X_1, \cdots\cdots X_{k_s};\ V_{k_1+\cdots+k_s+1}, \cdots\cdots, V_k)=1$$

である. ここで, $k_1+\cdots\cdots+k_s=l$ とする. $k-l=m$ とす. 故に, $U_1, \cdots\cdots U_{k_1}, W_1, \cdots\cdots W_{k_2}, \cdots\cdots, X_1, \cdots\cdots, X_{k_s}, V_{l+1}, \cdots\cdots, V_k$ の同時密度函数は,

$$\frac{\Gamma_p\left(\frac{n}{2}\right)}{\prod_{i=1}^{k+1}\Gamma_p\left(\frac{n_i}{2}\right)}|U_1|^{1/2(n_1-p-1)}\,|U_2-U_1|^{1/2(n_2-p-1)}\cdots\cdots|U_{k_1}-U_{k_1-1}|^{1/2(n_{k_1}-p-1)}$$
$$\cdot|W_1|^{1/2(n_{k_1+1}-p-1)}\,|W_2-W_1|^{1/2(n_{k_1+2}-p-1)}\cdots\cdots|W_{k_2}-W_{k_2-1}|^{1/2(n_{k_1+k_2}-p-1)}$$
$$\cdots\cdots\cdots\qquad\cdots\cdots\cdots$$
$$\cdot|X_1|^{1/2(n_{k_1+\cdots+k_{s-1}+1}-p-1)}\,|X_2-X_1|^{1/2(n_{k_1+\cdots+k_{s-1}+2}-p-1)}$$
$$\cdots\cdots|X_{k_s}-X_{k_s-1}|^{1/2(n_l-p-1)}$$
$$\cdot\prod_{i=l+1}^{k}|V_i|^{1/2(n_i-p-1)}\,\Big|I-(U_{k_1}+W_{k_2}+\cdots\cdots+X_{k_s})-\sum_{i=l+1}^{k}V_i\Big|^{1/2(n_{k+1}-p-1)}$$

となる.

$U_1, \cdots\cdots, U_{k_1-1}$ について積分すれば,

$$\frac{\prod_{i=1}^{k_1}\Gamma_p\left(\frac{n_i}{2}\right)}{\Gamma_p\left(\frac{1}{2}\sum_{i=1}^{k_1}n_i\right)}|U_{k_1}|^{1/2\left(\sum_{i=1}^{k_1}n_i-p-1\right)}.$$

$W_1, \cdots\cdots, W_{k_2-1}$ について積分すると,

$$\frac{\prod_{i=1}^{k_2}\Gamma_p\left(\frac{n_{k_1+i}}{2}\right)}{\Gamma_p\left(\frac{1}{2}\sum_{i=1}^{k_2}n_{k_1+i}\right)}|W_{k_2}|^{1/2\left(\sum_{i=1}^{k_2}n_{k_1+i}-p-1\right)},$$

$$\cdots\cdots\cdots\qquad\cdots\cdots\cdots$$

$X_1, \cdots, X_{k_s-1}$ について積分すると,

$$\frac{\prod_{i=1}^{k_s} \Gamma_p\left(\frac{n_{k_1+\cdots+k_{s-1}+i}}{2}\right)}{\Gamma_p\left(\frac{1}{2}\sum_{i=1}^{k_s} n_{k_1+\cdots+k_{s-1}+i}\right)} |X_{k_s}|^{1/2\left(\sum_{i=1}^{k_s} n_{k_1+\cdots+k_{s-1}+i}-p-1\right)}.$$

$V_{l+1}, \cdots, V_k$ について積分する為に,

$$R_j=(I-U_{k_1}-W_{k_2}-\cdots-X_{k_s})^{-1/2} V_j (I-U_{k_1}-W_{k_2}-\cdots-X_{k_s})^{-1/2},$$
$$(j=l+1, \cdots, k)$$

とすると,

$$J(V_{l+1}, \cdots, V_k \to R_{l+1}, \cdots, R_k)=|I-(U_{k_1}+W_{k_2}+\cdots+X_{k_s})|^{1/2(p+1)(k-l)}.$$

故に,

$$\prod_{j=l+1}^{k} |V_j|^{1/2(n_j-p-1)} \left|I-(U_{k_1}+W_{k_2}+\cdots+X_{k_s})-\sum_{j=l+1}^{k} V_j\right|^{1/2(n_{k+1}-p-1)}$$
$$=\prod_{j=l+1}^{k} |R_j|^{1/2(n_j-p-1)} \left|I-\sum_{j=l+1}^{k} R_j\right|^{1/2(n_{k+1}-p-1)}$$
$$\cdot |I-(U_{k_1}+\cdots+X_{k_s})|^{1/2\left(\sum_{j=l+1}^{k+1} n_j-p-1\right)}.$$

ここで, $R_{l+1}, \cdots, R_k$ について積分すれば,

$$\frac{\prod_{j=l+1}^{k+1} \Gamma_p\left(\frac{n_j}{2}\right)}{\Gamma_p\left(\frac{1}{2}\sum_{j=l+1}^{k+1} n_j\right)} |I-(U_{k_1}+W_{k_2}+\cdots+X_{k_s})|^{1/2\left(\sum_{j=l+1}^{k+1} n_j-p-1\right)}.$$

以上をまとめると,

$$\frac{\Gamma_p\left(\frac{n}{2}\right)}{\Gamma_p\left(\frac{n_1+\cdots+n_{k_1}}{2}\right)\cdots\Gamma_p\left(\frac{n_{k_1+\cdots+k_{s-1}}+\cdots+n_l}{2}\right)\Gamma_p\left(\frac{n_{l+1}+\cdots+n_{k+1}}{2}\right)}$$

$$\cdot |U_{k_1}|^{1/2\left(\sum_{i=1}^{s} n_i-p-1\right)} |W_{k_2}|^{1/2\left(\sum_{j=1}^{k_2} n_{k_1+j}-p-1\right)} \cdots |X_{k_s}|^{1/2\left(\sum_{j=1}^{k_s} n_{k_1+\cdots+k_{s-1}+j}-p-1\right)}$$

$$\cdot |I-(U_{k_1}+W_{k_2}+\cdots+X_{k_s})|^{1/2\left(\sum_{j=l+1}^{k+1} n_j-p-1\right)}.$$

故に, $U_{k_1}=Z_1$, $W_{k_2}=Z_2, \cdots,$ $X_{k_s}=Z_s$, とおけば,

$$\frac{\Gamma_p\left(\frac{n}{2}\right)}{\prod_{i=1}^{s+1} \Gamma_p\left(\frac{n^{(i)}}{2}\right)} \prod_{i=1}^{s} |Z_i|^{1/2(n^{(i)}-p-1)} \left|I-\sum_{i=1}^{s} Z_i\right|^{1/2(n^{(s+1)}-p-1)},$$

[定理 6]

$(U_1, \cdots, U_k) \in \text{ord-}D(n_1, \cdots, n_k; n_{k+1})$ とするとき，$(U_{k_1}, U_{k_1+k_2}, \cdots, U_{k_1+\cdots+k_s}) \in \text{ord-}D(n^{(1)}, \cdots, n^{(s)}; n^{(s+1)})$ となる．

(証明)

$$U_1 = V_1,\ U_2 = V_1 + V_2, \cdots,\ U_k = V_1 + V_2 + \cdots + V_k$$

とおくとき，$(V_1, \cdots, V_k) \in D(n_1, \cdots, n_k; n_{k+1})$ となる．

$$\begin{aligned} &U_{k_1} = V_1 + \cdots + V_{k_1},\ U_{k_1+k_2} = V_1 + \cdots + V_{k_1} + V_{k_1+1} + \cdots + V_{k_1+k_2}, \\ &\cdots, U_{k_1+\cdots+k_s} = V_1 + \cdots + V_{k_1} + V_{k_1+1} + \cdots + V_{k_1+k_2} + \cdots \\ &\qquad + V_{k_1+\cdots+k_{s-1}+1} + \cdots + V_{k_1+\cdots+k_s} \end{aligned}$$

であるから，定理5より，

$$U_{k_1} = Z_1,\ U_{k_1+k_2} = Z_1 + Z_2, \cdots,\ U_{k_1+\cdots+k_s} = Z_1 + \cdots + Z_s.$$

で，$(Z_1, \cdots, Z_s) \in D(n^{(1)}, n^{(2)}, \cdots, n^{(s)}; n^{(s+1)})$ となる．故に，$(U_{k_1}, U_{k_1+k_2}, \cdots, U_{k_1+\cdots+k_s}) \in \textbf{ord-}D(n^{(1)}, n^{(2)}, \cdots, n^{(s)}; n^{(s+1)})$．

[定理 7]

$(U_1, \cdots, U_k) \in \text{ord-}D(n_1, \cdots, n_{k+1})$ とするとき，

$$Pr\{X > U_k\} = \frac{\Gamma_p\left(\frac{n}{2}\right)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{1}{2}\sum_{j=1}^{k} n_j + \frac{1}{2}(p+1)\right)\Gamma_p\left(\frac{n_{k+1}}{2}\right)} |X|^{1/2 \sum_{j=1}^{k} n_j}$$

$$ {}_2F_1\left(\frac{1}{2}\sum_{j=1}^{k} n_j,\ \frac{-n_{k+1}+p+1}{2};\ \frac{\sum_{j=1}^{k} n_j + p + 1}{2};\ X\right).$$

(証明)

$$\begin{aligned} Pr\{X > U_k\} &= Pr\{X > U_k > \cdots > U_1 > 0\} \\ &= \frac{\Gamma_p\left(\frac{n}{2}\right)}{\prod_{i=1}^{k+1}\Gamma_p\left(\frac{n_i}{2}\right)} \int_0^X \int_0^{U_{k-1}} \cdots \int_0^{U_2} |U_1|^{1/2(n_1-p-1)}\, |U_2 - U_1|^{1/2(n_2-p-1)} \\ &\quad \cdots |U_k - U_{k-1}|^{1/2(n_k-p-1)} \cdot |I - U_k|^{1/2(n_{k+1}-p-1)}\, dU_1 \cdots dU_k. \end{aligned}$$

$U_1, U_2, \cdots U_{k-1}$ について積分すると，

$$\frac{\Gamma_p\left(\frac{n}{2}\right)}{\Gamma_p\left(\frac{1}{2}\sum_{j=1}^{k} n_j\right)\Gamma_p\left(\frac{n_{k+1}}{2}\right)} \int_0^X |U_k|^{1/2\left(\sum_{j=1}^{k} n_j - p - 1\right)}\, |I - U_k|^{1/2(n_{k+1}-p-1)}\, dU_k.$$

一方

$$\int_0^X |U|^{1/2(m-p-1)}\, |I-U|^{1/2(n-p-1)}\, dU$$

$$= \frac{\Gamma_p\left(\frac{m}{2}\right)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{m+p+1}{2}\right)} |X|^{m/2}\, {}_2F_1\left(\frac{m}{2},\ -\frac{n}{2} + \frac{p+1}{2};\ \frac{m+p+1}{2};\ X\right)$$

であるから，$\frac{1}{2}m = \frac{1}{2}\sum_{j=1}^{k} n_j$，$\frac{1}{2}n = \frac{1}{2}n_{k+1}$ とおけば得られる．

[定理 8]

$X_1 \in M\text{-}Be\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$, $X_2 \in M\text{-}Be\left(\frac{m_1}{2}, \frac{m_2}{2}\right)$ とし，互に独立とするとき，

$$Y = X_1^{1/2} X_2 X_1^{1/2}$$

の密度函数は

$$\frac{\Gamma_p\left(\frac{n_1+n_2}{2}\right)\Gamma_p\left(\frac{m_1+m_2}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{m_1}{2}\right)\Gamma_p\left(\frac{n_2+m_2}{2}\right)} |Y|^{1/2(m_1-p-1)} |I-Y|^{1/2(n_2+m_2-p-1)}$$

$$ {}_2F_1\left(\frac{n_2}{2}, \frac{m_1+m_2-n_1}{2}; \frac{n_2+m_2}{2}; I-Y\right).$$

(証明)

X_1 と X_2 が独立であるから，同時密度函数は

$$\frac{\Gamma_p\left(\frac{n_1+n_2}{2}\right)\Gamma_p\left(\frac{m_1+m_2}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m_1}{2}\right)\Gamma_p\left(\frac{m_2}{2}\right)} |X_1|^{1/2(n_1-p-1)} |X_2|^{1/2(m_1-p-1)}$$

$$|I-X_1|^{1/2(n_2-p-1)} |I-X_2|^{1/2(m_2-p-1)}$$

となる．

$X_2 = X_1^{-1/2} Y X_1^{-1/2}$ より，

$$J(X_1, X_2 \to X_1, Y) = |X_1|^{-1/2(p+1)}$$

であるから，

$$|X_1|^{1/2(n_1-p-1)} |X_2|^{1/2(m_1-p-1)} |I-X_1|^{1/2(n_2-p-1)} |I-X_2|^{1/2(m_2-p-1)}$$
$$= |X_1|^{1/2(n_1-m_1-m_2)} |I-X_1|^{1/2(n_2-p-1)} |X_1-Y|^{1/2(m_2-p-1)} |Y|^{1/2(m_1-p-1)}$$

$0<Y<X_1<I$. ここで X_1 について積分する必要が有る．$Z=I-X_1$ とおくと，

$$I-Y>Z>0, \quad J(X_1 \to Z) = 1.$$

より，積分すべき式は，

$$\int_0^{I-Y} |I-Z|^{1/2(n_1-m_1-m_2)} |Z|^{1/2(n_2-p-1)} |I-Y-Z|^{1/2(m_2-p-1)} dZ.$$

ここで，$|I-Z|^{1/2(n_1-m_1-m_2)}$ を Zonal 多項式で展開すると，

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\left(\frac{m_1+m_2-n_1}{2}\right)_\kappa \int_0^{I-Y} |Z|^{1/2(n_2-p-1)} |(I-Y)-Z|^{1/2(m_2-p-1)} C_\kappa(Z)\, dZ$$

$$= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\left(\frac{m_1+m_2-n_1}{2}\right)_\kappa |I-Y|^{1/2(n_2+m_2-p-1)}$$

$$\frac{\Gamma_p\left(\frac{n_2}{2};\kappa\right)\Gamma_p\left(\frac{m_2}{2}\right)}{\Gamma_p\left(\frac{n_2+m_2}{2};\kappa\right)} C_\kappa(I-Y).$$

以上より，Y の密度函数は，

$$\frac{\Gamma_p\left(\frac{n_1+n_2}{2}\right)\Gamma_p\left(\frac{m_1+m_2}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{m_1}{2}\right)\Gamma_p\left(\frac{n_2+m_2}{2}\right)} |Y|^{1/2(m_1-p-1)} |I-Y|^{1/2(n_2+m_2-p-1)}$$

$$\cdot {}_2F_1\left(\frac{n_2}{2},\ \frac{m_1+m_2-n_1}{2};\ \frac{n_2+m_2}{2};\ I-Y\right)$$

となる.

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