An estimating equation for modulated renewal processes

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1 Introduction

Consider a point process observed over time $t \geq 0$, and suppose that events occur at time $0 < t_1 < t_2 < \ldots$. Let $N(s,t)$ denotes the number of events occurring in time interval $[s,t)$, and $\mathscr{H}_t$ denotes the history of the process at time t. We also write $N(t)$ for $N(0,t)$. In general, a model for such process are specified by its conditional intensity as follows (e.g. see Cox and Isham, 1980, page 9):

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{\Pr\{N(t, t+\Delta t) = 1 \,|\, \mathscr{H}_t\}}{\Delta t}\,.$$

If the conditional intensity is expressed in the form of

$$\lambda(t) = \mu(t)\, Z_\beta(t)\,, \tag{1.1}$$

where $\mu(t) = \mu_0(t - t_{N(t)})$ is an unspecified baseline hazard function depending only on time t, and $Z_\beta(t)$ is a time-varying function with parameter β, then $N(t)$ is considered as a modulated renewal process. In particular, when $Z_\beta(t) = 1$, $N(t)$ reduces to a renewal process, i.e. the time differences between the consecutive events, $\delta_i = t_i - t_{i-1}$, for $i = 1, \ldots, n$, are independently and identically distributed.

2 Estimation of $\mu_0(x)$

Let $N(t)$ be a modulated renewal process with occurrence times $t_1, \ldots, t_n$ and conditional intensity (1.1). Assume that $t_0 = 0$ and $t_n = T$. According to the martingale property of the conditional intensity, for any predictable process $\xi(t) \geq 0$, $0 \leq t \leq T$,

$$\boldsymbol{E}\left[\int_0^T \xi(t)\, dt - \int_0^T \xi(t)\lambda(t)\, dt\right] = 0\,. \tag{2.1}$$

Here, we assume that $\xi(t)$ is a function of past events up to time t. Then, let

$$\xi(t) \;=\; \frac{\mu(t)}{\lambda(t)} \;=\; \frac{1}{Z_\beta(t)}\,,$$

and using the numerical integrations to write

$$\int_0^T \xi(t)\, dt \;\approx\; \sum_{i=1}^{n} \xi(t_i) \approx \int_0^T \mu(t)\, dt\,,$$

we derive an estimate for cumulative baseline intensity $U_0(x) = \int_0^x \mu_0(s)\, ds$:

$$\hat{U}_0(x) = \frac{\sum_{i=1}^n [\, \xi(t_i)\, \mathbf{1}(\delta_i < x) - U_0(\delta_i)\mathbf{1}(\delta_i < x)\,]}{\sum_{i=1}^n \mathbf{1}(\delta_i \geq x)}, \tag{2.2}$$

where $\mathbf{1}(\cdot)$ is an indicator function.

Next, we use penalized regression splines with B-spline basis functions of O'Sullivan (1986) to model data (δ_i, U_i), where $U_i = \hat{U}_0(\delta_i)$, $i = 1, \ldots, n$. The estimator for $U_0(x)$ is the minimizer of

$$\sum_{i=1}^n \left\{ U_i - \sum_{j=-p}^m \alpha_j B_{j,p+1}(\delta_i) \right\} + \omega \int_0^\Delta \left[\left\{ \sum_{j=-p}^m \alpha_j B_{j,p+1}(s) \right\}^{(q)} \right]^2 ds$$

where $B_{j,p+1}$ is the jth $(p+1)$-order B-spline basis function with a sequence of knots $0 = \kappa_{-p} = \ldots = \kappa_0 < \ldots < \kappa_{m+1} = \ldots = \kappa_{m+p+1} = \Delta = \max(\delta_i)$, ω is a smoothing parameter and the penalty is an integrated squared qth order derivative of the spline function. Then, the estimate of $\mu_0(x)$ is

$$\hat{\mu}_0(x) = \sum_{j=-p+1}^m \frac{p(\hat{\alpha}_j - \hat{\alpha}_{j-1})}{\kappa_{j+p} - \kappa_j} B_{j,p+1-q}(x), \quad \kappa_j \leq x < \kappa_{j+1},$$

where $\hat{\alpha}_j$ is a least-squares estimator of α_j.

3 Estimation of β

The log-likelihood of a point process is

$$\log L = -\int_0^T \lambda(t)\, dt + \sum_{i=1}^n \log \lambda(t_i).$$

Then, the maximum likelihood estimates of parameter β of model (1.1) is the maximizer of

$$\log L = \sum_{i=1}^n \left\{ -\int_0^{\delta_i} \mu_0(s)\, Z_\beta(s + t_{i-1})\, ds + \log \mu_0(\delta_i) + \log Z_\beta(t_i) \right\}.$$

4 Application

The illustrated method is applied to the occurrence times of the aftershocks of the great Sichuan (China) earthquake, with magnitude greater than 4.0, from May 12 to June 25, 2008.

References

Cox, D. R. and Isham, V.(1980). *Point Processes*, Chapman and Hall, London.

O'Sullivan, F.(1986). An statistical perspective on ill-posed inverse problems (with discussion), *Statistical Science*, **1**, 505–527.