Cosine perturbation of bimodal circular distributions

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1 Introduction

The von Mises distribution is the well-known circular distribution in *Directional Statistics*, and its probability density function (pdf) is given by

$$f_0(\theta;\mu,\kappa)=\frac{e^{\kappa\cos(\theta-\mu)}}{2\pi I_0(\kappa)},\quad -\pi\le\theta<\pi, \tag{1}$$

where $-\pi\le\mu<\pi$, $\kappa\ge 0$ and $I_r(\kappa)$ is the modified Bessel function of the first kind of order r, defined as

$$I_r(\kappa)=\frac{1}{2\pi}\int_0^{2\pi}\cos(r\theta)\,e^{\kappa\cos\theta}d\theta,\quad r=0,\pm1,\pm2,\ldots.$$

The pdf of the GVM distribution which can also be used to model asymmetry and bimodality is given by

$$f(\theta)\propto\exp\{\kappa_1\cos(\theta-\mu_1)+\kappa_2\cos 2(\theta-\mu_2)\},\quad -\pi\le\theta<\pi, \tag{2}$$

where $-\pi\le\mu_1<\pi$, $-\pi/2\le\mu_2<\pi/2$ and $\kappa_1,\kappa_2\ge 0$.

The mixture of two von Mises (MVM) distributions has pdf

$$f(\theta)=pf_0(\theta;\mu_1,\kappa_1)+(1-p)f_0(\theta;\mu_2,\kappa_2),\quad -\pi\le\theta<\pi,$$

where $-\pi\le\mu_1,\mu_2<\pi$, $\kappa_1,\kappa_2\ge 0$ and p $(0\le p\le 1)$ is the mixing parameter. The most simple way to construct a bimodal circular distribution is perhaps two times a circular random variable which has a unimodal symmetric circular distribution whose domain is restricted to $[-\pi,\pi)$. But it is rare that the two modes of the distribution have the same concentration. To make different concentrations about the main and sub modes to the bimodal circular distribution, we consider the following construction.

Theorem 1.1 *Suppose that f_1 and g are circular pdf which are symmetric about the the zero direction, and $G(\theta)=\int_{-\pi}^{\theta}g(\phi)d\phi$ is the distribution function corresponding to the latter. Let the weighting function $u(\theta)$ be an even function and periodic, i.e., $u(-\theta)=u(\theta)$ and $u(\theta)=u(\theta+2\pi k)$ for all integers k, such that $u(\theta)=-u(\pi-\theta)$ and $|u(\theta)|\le\pi$. If the function $f_1(\theta)$ also satisfies $f_1(\theta)=f_1(\pi-\theta)$ for $\theta\in[0,\pi)$, then*

$$f(\theta)=2G(u(\theta))f_1(\theta),\quad \theta\in[-\pi,\pi) \tag{3}$$

is a symmetric circular pdf.

An extension with location parameter μ $(-\pi\le\mu<\pi)$ is defined by $\theta\mapsto\theta-\mu$ in (3).

Let $f_0(\theta)$ be a circular pdf defined on $[-\pi,\pi)$ and symmetric about $\theta=0$. Then the function $f_0(2\theta)$ is also a circular pdf defined on $[-\pi,\pi)$ which satisfies the conditions of Theorem 1.1. Let G be the distribution function of the uniform distribution, i.e., $G(\theta)=(\pi+\theta)/2\pi$. Introducing a parameter λ (initially, with $-1\le\lambda\le 1$), let $u(\theta)=\lambda\pi\cos\theta$ for $\theta\in[-\pi,\pi)$ in (3). Then we have a cosine perturbed family of bimodal circular distributions with pdf

$$f(\theta)=(1+\lambda\cos\theta)f_0(2\theta),\quad -\pi\le\theta<\pi. \tag{4}$$

Since $f(\theta;-\lambda)=(1-\lambda\cos\theta)f_0(2\theta)=\{1+\lambda\cos(\theta+\pi)\}f_0(2(\theta+\pi))=f(\theta+\pi;\lambda)$ if $-\pi\le\theta\le 0$ and $f(\theta;-\lambda)=f(\theta-\pi;\lambda)$ if $0\le\theta<\pi$, the range of the parameter λ can be further constrained as $0\le\lambda\le 1$. The parameter λ plays a role of a weight for the directions at $\theta=0$ and $-\pi$. The pdf with (4) is unperturbed when $\lambda=0$, otherwise the function $f(\theta)$ always takes the maximum at $\theta=0$ and it is more concentrated at $\theta=0$ than at $\theta=-\pi$.

2 Cosine perturbed bimodal Jones–Pewsey distribution

Among the symmetric distributions on the circle, the Jones–Pewsey distribution (Jones and Pewsey, 2005), which includes the von Mises, cardioid and wrapped Cauchy distributions, is a reasonable candidate as a flexible model for the circular data. In this section, we adopt the Jones–Pewsey distribution as a base model, and propose the cosine perturbed bimodal Jones–Pewsey (CPBJP) distribution. The density of the Jones–Pewsey family can be represented as

$$f_0(\theta)=\frac{\{\cosh(\kappa\psi)+\sinh(\kappa\psi)\cos\theta\}^{1/\psi}}{2\pi P_{1/\psi}(\cosh(\kappa\psi))},\quad -\pi\le\theta<\pi,$$

where $\kappa\ge 0$, $-\infty<\psi<\infty$ and $P_{1/\psi}(\cdot)$ is the associated Legendre function of the first kind of degree $1/\psi$ and order 0. Employing this density for $f_0(2\theta)$ in (4), the density of the CPBJP distribution can be represented as

$$f(\theta)=\frac{\cosh^{1/\psi}(\kappa\psi)(1+\lambda\cos\theta)\{1+\tanh(\kappa\psi)\cos(2\theta)\}^{1/\psi}}{2\pi P_{1/\psi}(\cosh(\kappa\psi))},\quad -\pi\le\theta<\pi,$$

where $0\le\lambda\le 1$.

3 Parameter estimation and significance testing

In this section we consider parameter estimation for a random sample of size n, $\theta_1,\ldots,\theta_n$, drawn from the cosine perturbed distribution with pdf (4).

3.1 Maximum likelihood

Suppose that the density $f_0(2(\theta-\eta),\mathbf{t})$ depends on the vector-valued parameter $\mathbf{t}=(t_1,t_2,\ldots,t_l)$ as well as the location parameter η. Then the log-likelihood function, $\ell(\lambda,\eta,\mathbf{t})$, of its cosine perturbed bimodal counterpart with density (4) can be represented as

$$\begin{aligned}\ell(\lambda,\eta,\mathbf{t})&=\sum_{i=1}^{n}\log\{1+\lambda\cos(\theta_i-\eta)\}+\ell(0,\eta,\mathbf{t})\\&=\sum_{i=1}^{n}\log\{1+\lambda\cos(\theta_i-\eta)\}+\sum_{i=1}^{n}\log f_0(2(\theta_i-\eta),\mathbf{t}).\end{aligned}$$

An important point of the distribution with pdf (4) is 'unidentifiable' when $\lambda=0$. In that case, the location parameter η will be constrained as $-\pi/2\le\eta<\pi/2$ to avoid the problem.

3.2 Test of equal concentration

To investigate whether the model (4) has the two equal concentrations, we test $H_0:\lambda=0$ against $H_1:\lambda>0$. The likelihood ratio test gives the test statistic as

$$T=-2\frac{\max_{\eta,\mathbf{t}}\ell(0,\eta,\mathbf{t})}{\max_{\lambda,\eta,\mathbf{t}}\ell(\lambda,\eta,\mathbf{t})}.$$

The asymptotic distribution of T is $Z^2I[Z>0]$, where Z is distributed as the standard normal and I denotes an indicator function defined by $I[Z>0]=1$ if $Z>0$, $=0$ if $Z\le 0$. This means that the asymptotic distribution is a half-half mixture of χ_0^2 and χ_1^2.

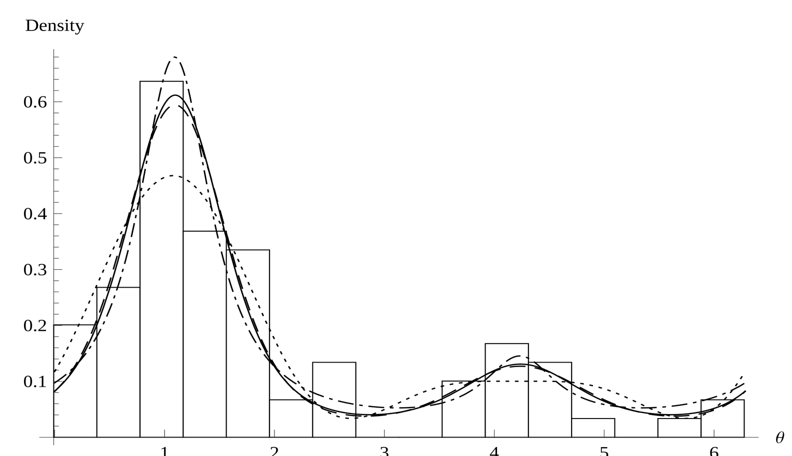
4 Illustrative example

4.1 The turtle data

As our example we consider the turtle data in Appendix B.3 of Fisher (1993). The data consist of the directions chosen by 76 female turtles after egg-laying on a beach.

Table presents the result of MLEs, and the corresponding MLL, AIC and p-value of the chi-squared goodness-of-fit test of the four cosine perturbed bimodal models considered in Section 2. Figure portrays a histogram for 76 turtles, together with the maximum likelihood fits for the CPBVM (dash), CPBC (dot), CPBWC (dash-dot) and CPBJP (solid) distributions. One of the remarkable points is that all the values of $\hat{\lambda}$ are almost the same value (actually, they differ in the level of three decimal places). Histogram for 76 turtles, together with the maximum likelihood fits for the CPBVM (dash), CPBC (dot), CPBWC (dash-dot) and CPBJP (solid) distributions. The horizontal axis runs from 0 to 2π.

Distn	$\hat{\mu}$	$\hat{\kappa}$	$\hat{\lambda}$	$\hat{\psi}$	MLL	AIC	p (g-o-f)
CPBJP	1.10	1.08	0.65	−0.16	−107.58	223.16	0.24
CPBVM	1.10	1.10	0.65		−107.62	221.24	0.44
CPBC	1.08	1.06	0.65		−110.01	226.01	0.43
CPBWC	1.09	0.95	0.65		−108.33	222.67	0.21



Finally, since the p-value of the test of equal concentration described in Section 3.2 is 0.0000 with log-likelihood ratio $2(-107.57+120.88)=26.61$ when we use the CPBJP as a model, therefore the null hypothesis $H_0:\lambda=0$ is rejected at the 5% significance level. Hence, the turtles significantly prefer the direction at $\hat{\mu}$ to the direction at $\hat{\mu}+\pi$ from the test. The same result is derived by the other cosine perturbed distributions.

References

[1] Fisher, N.I. (1993). *Statistical Analysis of Circular Data*. Cambridge University Press, Cambridge.

[2] Jones, M.C. and Pewsey, A. (2005). A family of symmetric distributions on the circle. *J. Amer. Statist. Assoc.*, **100**, 1422–1428.