Self-dual polyhedral cones

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Based on a joint work [2] with João Gouveia (University of Coimbra).

1 Polyhedral cones

$\mathcal{K} \subseteq \mathbb{R}^d$ is a **polyhedral cone** $\overset{\text{def}}{\iff}$ $\mathcal{K}$ is the set of solutions of finitely many linear inequalities, i.e., $\exists A \in \mathbb{R}^{n\times d}$ such that $\mathcal{K} = \{x \in \mathbb{R}^d \mid Ax \geq 0\}$.

Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^d$. The **dual cone** is

$$\mathcal{K}^* := \{y \in \mathbb{R}^d \mid \langle x, y\rangle \geq 0, \forall x \in \mathcal{K}\}.$$

$\mathcal{K}$ is **self-dual** $\overset{\text{def}}{\iff}$ there exists *some* inner product such that $\mathcal{K} = \mathcal{K}^*$.

When is $\mathcal{K}$ a self-dual cone?

Since $\langle \cdot, \cdot \rangle$ may be arbitrary, hard to answer in general, e.g., see [3].

2 Slack matrices

A **slack matrix** [4, 1] $M \in \mathbb{R}^{n\times m}$ of $\mathcal{K}$ is matrix obtained as follows:

- Let $\{u_1 \ldots, u_n\}$ be the extreme rays of $\mathcal{K}$.
- Let $\{v_1, \ldots, v_m\}$ be the extreme rays of $\mathcal{K}^*$.
- $M_{ij} := \langle u_i, v_j\rangle$.

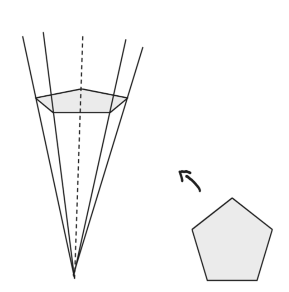
Why? Structural properties of $\mathcal{K}$ $\Leftrightarrow$ linear algebraic properties of M.

Example 1: $\mathbb{R}^3_+$

- **Extreme rays**: $\{e_1, e_2, e_3\}$.
- **Slack matrices**: $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1\end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix}$, etc.

Example 2: cone over a pentagon Start with a regular pentagon P_5 in $\mathbb{R}^2$ centered at the origin and take the cone in $\mathbb{R}^3$ generated by $1 \times P_5$.

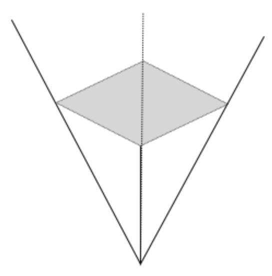
- **Extreme rays**: $(\cos(2\pi i/5), \sin(2\pi i/5), \sqrt{-\cos(4\pi/5)}), i = 0, \ldots, 4$.
- **Example of slack matrix**:



$$\begin{pmatrix} 1+\cos(\frac{\pi}{5}) & \frac{\sqrt{5}}{2} & 0 & 0 & \frac{\sqrt{5}}{2} \\ \frac{\sqrt{5}}{2} & 1+\cos(\frac{\pi}{5}) & \frac{\sqrt{5}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{5}}{2} & 1+\cos(\frac{\pi}{5}) & \frac{\sqrt{5}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{5}}{2} & 1+\cos(\frac{\pi}{5}) & \frac{\sqrt{5}}{2} \\ \frac{\sqrt{5}}{2} & 0 & 0 & \frac{\sqrt{5}}{2} & 1+\cos(\frac{\pi}{5}) \end{pmatrix}$$

Example 3: 1-norm cone $\mathcal{L}^3_1 = \{(x, y, z) \in \mathbb{R}^3 \mid |x| + |y| \leq z\}$

- **Extreme rays**: $\{(1, 0, 1), (-1, 0, 1), (0, 1, 1), (0, -1, 1)\}$.
- **Extreme rays of** $(\mathcal{L}^3_1)^*$: $\{(1, 1, 1), (-1, 1, 1), (1, -1, 1), (-1, -1, 1)\}$.
- **Slack matrices**: $\begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2\end{pmatrix}$, etc.



3 An expected result

Theorem 1. *The following are equivalent.*

(i) $\mathcal{K}$ is self-dual with respect to some inner product.

(ii) Every slack matrix of $\mathcal{K}$ is either PSD or becomes PSD after permuting rows and rescaling columns.

4 A surprising result

PSD slack matrices are nonnegative so they are doubly nonnegative matrices.

Theorem 2. *If $\mathcal{K} \subseteq \mathbb{R}^d$ is self-dual and irreducible and $M \in \mathbb{R}^{n\times n}$ is a PSD slack of $\mathcal{K}$, then M is a rank d extreme ray of DNN^n.*

Theorem 3. *$X \in \mathrm{DNN}^5$ is an extreme ray if and only if X is rank 1 or X is the slack matrix of an irreducible cone over a pentagon.*

5 A weird result

We recall some definitions.

- $X \in \mathrm{CP}^n \overset{\text{def}}{\iff} \exists V \in \mathbb{R}^{n\times m}$, with $V \geq 0$ and $X = VV^T$
- $X \in \mathrm{CS}^n$ (completely positive semidefinite) $\overset{\text{def}}{\iff} \exists A_1, \ldots, A_n \in \mathcal{S}^m_+$ such that $X_{ij} = \mathrm{tr}\,(A_iA_j)$.
- $\mathrm{CP}^n = \mathrm{CS}^n$, for $n \leq 4$ and $\mathrm{CP}^n \subsetneq \mathrm{CS}^n$ for $n \geq 5$.

Theorem 4. *Let S be a PSD slack matrix of a self-dual polyhedral cone $\mathcal{K}$. If S is not diagonal, then $S \notin \mathrm{CS}^n$*

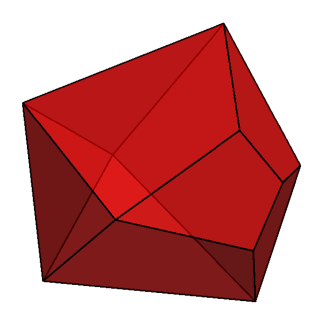
- $\mathcal{K} \subseteq \mathbb{R}^d$ has a diagonal slack $\Leftrightarrow$ $\mathcal{K}$ is linearly isomorphic to $\mathbb{R}^d_+$
- **Informal conclusion**: PSD slacks are typically neither completely positive nor completely positive semidefinite!

6 Finding self-dual cones via SDP

- Let M_{sup} be the **support** of a slack matrix of a combinatorially self-dual cone $\mathcal{K} \subseteq \mathbb{R}^d$.
- Permute the columns of M_{sup} so that M_{sup} is **symmetric** and has **no zeroes in the diagonal**.
  - If not possible, there is no self-dual cone whose slack matrix has the same support.
- Find $X \in \mathrm{DNN}^n$ with rank d such that $X_{ij} = 0$ if and only if $(M_{\mathrm{sup}})_{ij} = 0$.
- If $X = UU^T$, with $U \in \mathbb{R}^{n\times d}$, then $\mathcal{K}' := \mathrm{cone\,rows}(U)$ is self-dual.

$$\begin{aligned} \max_{X \succeq 0} \quad & \sum_{i,j} X_{i,j} \\ \text{s.t.} \quad & X_{i,j} = 0 \text{ if } (M_{\mathrm{sup}})_{i,j} = 0; \\ & X_{i,i} = 1 \text{ for all } i. \end{aligned}$$

- We relax the rank constraint and the non-negativity. Looks weak, but works surprisingly well.



A 3D slice of a 4D numerically self-dual cone.

参考文献

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