A Bayesian non-parametric method of estimating the background intensity of ETAS model

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1 Introduction

The spatio-temporal epidemic type aftershock sequence (ETAS) model, which is an example of a self-exciting, spatio-temporal, marked Hawkes process model, is widely used in statistical seismology. The background intensity is important for assessing seismic hazard and forecasting seismic activities (Molkenthin et al., 2022). This poster shows a Bayesian non-parametric way of estimating the background intensity.

Conditional intensity function

The ETAS model is characterized by its conditional intensity function,

$$\lambda(t, \boldsymbol{x} \mid \mathcal{H}_t, \boldsymbol{\theta}_\mu, \boldsymbol{\theta}_\varphi) = \mu(\boldsymbol{x} \mid \boldsymbol{\theta}_\mu) + \sum_{i:t_i<t} \varphi(t - t_i, \boldsymbol{x} - \boldsymbol{x}_i \mid m_i, \boldsymbol{\theta}_\varphi).$$

Latent branching structure

Each event i has a non observable latent random variable $z_i \in \{0, 1, \cdots, i-1\}$, where

$$z_i = \begin{cases} 0 & \text{event i is a background event} \\ j > 0 & \text{event i is triggered by event j} \end{cases}. \tag{1}$$

$$\begin{aligned} p_{i0} = p(z_i = 0) &= \frac{\mu(\boldsymbol{x}_i \mid \boldsymbol{\theta}_\mu)}{\lambda(t_i, \boldsymbol{x}_i \mid \mathcal{H}_{t_i}, \boldsymbol{\theta}_\mu, \boldsymbol{\theta}_\varphi)} \\ p_{ij} = p(z_i = j) &= \frac{\varphi(t_i - t_j, \boldsymbol{x}_i - \boldsymbol{x}_j \mid m_j, \boldsymbol{\theta}_\varphi)}{\lambda(t_i, \boldsymbol{x}_i \mid \mathcal{H}_{t_i}, \boldsymbol{\theta}_\mu, \boldsymbol{\theta}_\varphi)} \end{aligned} \tag{2}$$

with $p_{i0} + \sum_{j=1}^{i-1} p_{ij} = 1$.

2 Methodology

All the observations are $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_\varphi$, where $\mathcal{D}_0 = \{(t_i, x_i, m_i, z_i = 0)\}_{i=1}^{N_{\mathcal{D}_0}}$ are background events according to a Poisson process with intensity $\mu(\boldsymbol{x})$, and $\mathcal{D}_\varphi = \{(t_i, x_i, m_i, z_i \neq 0)\}_{i=1}^{N_{\mathcal{D}_\varphi}}$ are offspring events.

Priori

The background intensity is defined by

$$\mu(\boldsymbol{x}) = \bar{\lambda}\,\sigma(f(\boldsymbol{x})) = \bar{\lambda}\frac{1}{1 + e^{-f(\boldsymbol{x})}}, \tag{3}$$

where $\sigma(\cdot)$ is the logistic sigmoid function.

- $\bar{\lambda} \sim$ Gamma distribution.
- $f(\boldsymbol{x}) \sim \mathcal{GP}$ prior with zero mean and a covariance function

$$k(\boldsymbol{x}, \boldsymbol{x}' \mid \boldsymbol{v}) = v_0 \prod_{i=1}^{2} \exp\left(-\frac{(x_i - x_i')^2}{2v_i^2}\right). \tag{4}$$

- $\boldsymbol{v} \sim$ Exponential distribution.

Likelihood

- Likelihood of the latent branching structure is

$$p(\mathcal{D}, Z \mid \mu(\boldsymbol{x}), \boldsymbol{\theta}_\varphi) = \underbrace{\prod_{i=1}^{N_\mathcal{D}} \mu(\boldsymbol{x}_i)^{\mathbb{I}(z_i=0)} \exp\left(-|\mathcal{T}| \int_{\mathcal{X}} \mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right)}_{p(\mathcal{D}_0 | Z, \mu(\boldsymbol{x}))} \times\ p(\mathcal{D} \mid Z, \boldsymbol{\theta}_\varphi) \times p(Z).$$

- Given a branching structure Z, the background intensity is a Poisson likelihood with (3).

$$p(\mathcal{D}_0 \mid Z, f(\boldsymbol{x}), \bar{\lambda}) = \prod_{i=1, z_i=0}^{N_\mathcal{D}} \bar{\lambda}\,\sigma(f(\boldsymbol{x}_i)) \exp\left(-|\mathcal{T}| \int_{\mathcal{X}} \bar{\lambda}\,\sigma(f(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right).$$

- To reduce the intractable integral inside the exponential term to a constant, an independent latent Poisson process $\Pi = \{\boldsymbol{x}_l\}_{l=N_\mathcal{D}+1}^{N_{\mathcal{D}\cup\Pi}}$ with rate $\lambda(x) = \lambda(1 - \sigma(f(x)))$ is introduced.

$$p(\mathcal{D}_0, \Pi \mid Z, f(\boldsymbol{x}), \bar{\lambda}) = \prod_{i=1, z_i=0}^{N_\mathcal{D}} \bar{\lambda}\,\sigma(f(\boldsymbol{x}_i)) \prod_{l=N_\mathcal{D}+1}^{N_{\mathcal{D}\cup\Pi}} \bar{\lambda}\,\sigma(-f(\boldsymbol{x}_l)) \exp\left(-|\mathcal{T}||\mathcal{X}|\bar{\lambda}\right). \tag{5}$$

- To obtain a likelihood which has a Gaussian form and is conditionally conjugate to the $\mathcal{GP}$ prior denoted by $p(\boldsymbol{f})$, the Pólya–Gamma random variables are introduced (Polson et al., 2013).

$$p(\mathcal{D}_0, \Pi, \omega_\mathcal{D}, \omega_\Pi \mid Z, f(\boldsymbol{x}), \bar{\lambda}) = \prod_{i=1, z_i=0}^{N_\mathcal{D}} \frac{\bar{\lambda}}{2} e^{\frac{f_i}{2} - \frac{f_i^2}{2}\omega_i} p_{\mathrm{PG}}(\omega_i \mid 1, 0) \prod_{l=N_\mathcal{D}+1}^{N_{\mathcal{D}\cup\Pi}} \frac{\bar{\lambda}}{2} e^{-\frac{f_l}{2} - \frac{f_l^2}{2}\omega_l} p_{\mathrm{PG}}(\omega_l \mid 1, 0) \exp\left(-|\mathcal{T}||\mathcal{X}|\bar{\lambda}\right).$$

Conditional posteriors for the kth Gibbs iteration

- $Z_i^{(k)} \mid \mathcal{D}, \mu(\boldsymbol{x}_i)^{(k-1)}, \boldsymbol{\theta}_\varphi^{(k-1)} \sim \text{Categorical}(p_i)$
- $\Pi^{(k)} \mid \bar{\lambda}^{(k-1)}, \boldsymbol{f}^{(k-1)} \sim \mathrm{PP}(\bar{\lambda}(\sigma(-f(\boldsymbol{x}))))$
- $\forall i: z_i = 0, \quad \omega_i^{(k)} \mid f_l^{(k-1)}, \mathcal{D}, Z^{(k)} \sim p_{\mathrm{PG}}(\omega_i \mid 1, |f_i|)$
- $\forall l: N_\mathcal{D} + 1, \cdots, N_{\mathcal{D}\cup\Pi}, \quad \omega_l^{(k)} \mid f_l^{(k-1)}, \Pi^k \sim p_{\mathrm{PG}}(\omega_l \mid 1, |f_l|)$
- $\bar{\lambda}^{(k)} \mid Z^{(k)}, \Pi^k \sim \text{Gamma}(N_{\mathcal{D}\cup\Pi} + \alpha_0, |\mathcal{T}||\mathcal{X}| + \beta_0)$
- $\boldsymbol{f}^{(k)} \mid \mathcal{D}, \omega_\mathcal{D}^{(k)}, \omega_\Pi^{(k)}, \Pi^k, Z^{(k)} \sim$ Gaussian distribution

References

Molkenthin, C., Donner, C., Reich, S., Zöller, G., Hainzl, S., Holschneider, M., Opper, M., 2022. Gp-etas: semiparametric bayesian inference for the spatio-temporal epidemic type aftershock sequence model. Statistics and computing 32, 29.

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