Geometry of hyperbolicity cones

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1 Hyperbolic polynomial and hyperbolicity cones

Let $p : \mathbb{R}^n \to \mathbb{R}$ be a homogenous polynomial of degree d and $e \in \mathbb{R}^n$ such that $p(e) > 0$. If for every $x \in \mathbb{R}^n$

$$t \mapsto p(x - te)$$

has only real roots, then p is **hyperbolic** along e [1, 2, 7].

For $x \in \mathbb{R}^n$, denote the roots of $t \mapsto p(x - te)$ by $\lambda_1(x), \ldots, \lambda_d(x)$. Then, the corresponding **hyperbolicity cone** is

$$\Lambda_+(p, e) \coloneqq \{x \in \mathbb{R}^n \mid \lambda_i(x) \geq 0, i = 1, \ldots, d\}.$$

Examples:

- $p(x) \coloneqq x_1 \cdots x_n, \quad e \coloneqq (1, \ldots, 1) \Rightarrow \Lambda_+(p, e) = \mathbb{R}^n_+$
- $p(X) \coloneqq \det X,\ e \coloneqq I_n \Rightarrow \Lambda_+(p, e) = \mathcal{S}^n_+$
- All symmetric cones are hyperbolicity cones.

Spectrahedral cone: $\mathcal{K}$ is a spectrahedral cone $\overset{\text{def}}{\iff}$ there exists a linear subspace $L \subseteq \mathcal{S}^n$ such that $\mathcal{K} \cong \mathcal{S}^n_+ \cap L$.

Lax Conjecture: Is every hyperbolicity cone isomorphic to a spectrahedral cone (i.e., a slice of a positive semidefinite cone)?

1.1 Goals of this project

Study the **convex geometric properties** of hyperbolicity cones and spectrahedral cones.

2 Facial exposedness

Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex cone. We have the following definitions.

- $\mathcal{K}$ is **amenable** [4] $\overset{\text{def}}{\iff}$ for every face $\mathcal{F} \trianglelefteq \mathcal{K}$ there is $\kappa > 0$ such that
$$\operatorname{dist}(x, \mathcal{F}) \leq \kappa \operatorname{dist}(x, \mathcal{K}), \qquad \forall x \in \operatorname{span} \mathcal{F}.$$
- $\mathcal{K}$ is **facially exposed** $\overset{\text{def}}{\iff}$ for every face $\mathcal{F} \trianglelefteq \mathcal{K}$, there exists a supporting hyperplane H of $\mathcal{K}$ such that $\mathcal{F} = \mathcal{K} \cap H$.

Amenability is a **strictly stronger** property than facial exposedness because every amenable cone is facially exposed and there are examples of facial exposed cones that are not amenable.

Some new results:

- Every spectrahedral cone is amenable. [6]
- Every hyperbolicity cone is amenable. [5]

3 Rank-one generated hyperbolicity cones

Given $p : \mathbb{R}^n \to \mathbb{R}$ a hyperbolicity polynomial we define $\operatorname{rank}(x)$ as the number of nonzero eigenvalues of x. That is,

$$\operatorname{rank}(x) = \text{number of nonzero roots of } t \mapsto p(x - te).$$

From now, we suppose that $\Lambda_+(p, e)$ is **regular** (i.e., **pointed** and **full-dimensional**).

$\Lambda_+(p, e)$ is **rank-one generated (ROG)** [3] $\overset{\text{def}}{\iff}$ every extreme ray of $\Lambda_+(p, e)$ is generated by a rank 1 element.

Examples: all symmetric cones and all spectrahedral cones whose extreme rays correspond to rank 1 matrices.

4 Automorphisms of ROG hyperbolicity cones and their derivative relaxations

For simplicity, we denote $\Lambda_+(p, e)$ by Λ_+.

Renegar's derivative relaxations: The following characterization holds

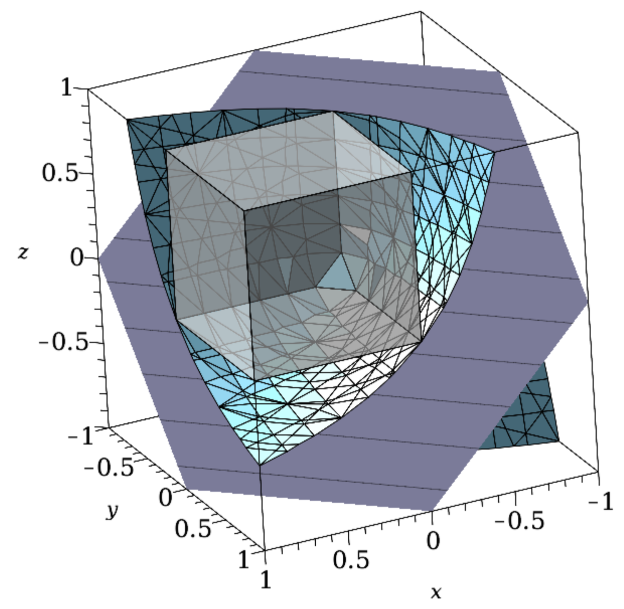
$$\Lambda_+ = \{x \in \mathbb{R}^n \mid p(x) \geq 0, D^1_e p(x) \geq 0, \ldots, D^{d-1}_e p(x) \geq 0\}.$$

The m-th order derivative relaxation of Λ_+ is

$$\Lambda_+^{(m)} = \{x \in \mathbb{R}^n \mid D^m_e p(x) \geq 0, \ldots, D^{d-1}_e p(x) \geq 0\}.$$

We have the following inclusions.

$$\Lambda_+ = \Lambda_+^{(0)} \subseteq \Lambda_+^{(1)} \subseteq \cdots \subseteq \Lambda_+^{(d-1)}$$



4.1 New results on automorphisms of ROG hyperbolicity cones

Theorem 4.1 ([3]). *Suppose that Λ_+ is regular and ROG. For $1 \leq k \leq \deg p - 3$ we have*

$$\operatorname{Aut}(\Lambda_+^{(k)}) = \{A \in \operatorname{Aut}(\Lambda_+) \mid A(\mathbb{R}_+ e) = \mathbb{R}_+ e\}.$$

Theorem 4.2 ([3]). *For $n \geq 4$ and k with $1 \leq k \leq n - 3$, we have*

$$\operatorname{Aut}(\mathbb{R}_+^{n,(k)}) = \{\alpha P \mid \alpha > 0,\ P \text{ is a permutation matrix}\}$$
$$\operatorname{Aut}(\mathcal{S}_+^{n,(k)}) = \{\alpha L_Q \mid \alpha > 0,\ Q \text{ is a } n \times n \text{ orthogonal matrix}\},$$

where $L_Q : \mathcal{S}^n \to \mathcal{S}^n$ is the map such that $L_Q(X) = QXQ^T$ holds for every $X \in \mathcal{S}^n$.

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