⑧ Wishart分布のDerivationについて

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(x, y, z) は正則なる次元正規分布

$$(2\pi)^{-3/2}\Lambda^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\left(\frac{\Lambda_{11}}{\Lambda}x^2+\frac{\Lambda_{22}}{\Lambda}y^2+\frac{\Lambda_{33}}{\Lambda}z^2+2\frac{\Lambda_{12}}{\Lambda}xy+2\frac{\Lambda_{13}}{\Lambda}xz+2\frac{\Lambda_{23}}{\Lambda}yz\right)\right] \tag{1}$$

に従うものとし，これからの任意標本

$$(x_i,\ y_i,\ z_i) \qquad i=1,2,\cdots\cdots,n$$

について

$$\begin{aligned} m_{11}&=\sum_1^n x_i^2, \quad m_{12}=\sum_1^n x_i y_i, \quad m_{13}=\sum_1^n x_i z_i \\ &\qquad m_{22}=\sum_1^n y_i^2, \quad m_{23}=\sum_1^n y_i z_i \\ &\qquad\qquad m_{33}=\sum_1^n z_i^2 \end{aligned} \tag{2}$$

の同時分布を求めようというのである。

(1) を書き直して

$$\sqrt{\frac{\Lambda_{33}}{2\pi\Lambda}}\exp\left[-\frac{1}{2}\left(\frac{\Lambda_{33}}{\Lambda}z^2+2\frac{\Lambda_{31}}{\Lambda}xz+2\frac{\Lambda_{32}}{\Lambda}yz+\left(\frac{\Lambda_{11}}{\Lambda}-\frac{\Lambda_{33\cdot11}}{\Lambda_{33}}\right)x^2+2\left(\frac{\Lambda_{12}}{\Lambda}-\frac{\Lambda_{33\cdot12}}{\Lambda_{33}}\right)xy+\left(\frac{\Lambda_{22}}{\Lambda}-\frac{\Lambda_{33\cdot22}}{\Lambda_{33}}\right)y^2\right)\right]dz$$

$$\times\sqrt{\frac{A_{33.22}}{2\pi A_{33}}}\ exp\left[-\frac{1}{2}\left(\frac{A_{33.22}}{A_{33}}y^2+2\frac{A_{33.12}}{A_{33}}xy+\left(\frac{A_{33.11}}{A_{33}}-\frac{1}{A_{33.22}}\right)x^2\right)\right]dy$$

$$\times\sqrt{\frac{1}{2\pi A_{33.22}}}\ exp\left(-\frac{x^2}{2A_{33.22}}\right)dx \qquad (3)$$

とする。　これを略記して

$$p_1(z\mid x.y)\,dz\ p_2(y\mid x)\,dy\ p_3(x)\,dx \qquad (4)$$

とおく。　(x_i, y_i, z_i)　$i=1, 2, \cdots, n$ の同時分布は

$$\prod_{i=1}^{n}p_1(z_i\mid x_i.y_i)\,dz_1\cdots dz_n\prod_{i=1}^{n}p_2(y_i\mid x_i)\,dy_1\cdots dy_n\prod_{i=1}^{n}p_3(x_i)\,dx_1\cdots dx_n \qquad (5)$$

である。

さて Conditional な $z_1\cdots z_n$ の分布を考えるなら

$$\left(\sqrt{\frac{A_{33}}{2\pi A}}\right)^n exp\left[-\frac{1}{2}\left\{\left(\frac{A_{11}}{A}-\frac{A_{33.11}}{A_{33}}\right)\sum_1^n x_i^2+2\left(\frac{A_{12}}{A}-\frac{A_{33.12}}{A_{33}}\right)\sum_1^n x_iy_i+\left(\frac{A_{22}}{A}-\frac{A_{33.22}}{A_{33}}\right)\sum_1^n y_i\right\}\right]$$

$$\times\ exp.\left[-\frac{1}{2}\left\{\frac{A_{33}}{A}\sum_1^n z_i^2+2\frac{A_{33}}{A}\sum_1^n x_iz_i+2\frac{A_{32}}{A}\sum_1^n y_iz_i\right\}\right]dz_1\cdots dz_n \qquad (6)$$

$X=(x_1\cdots x_n)$, $Y=(y_1\cdots y_n)$ を固定したときの $Z=(z_1\cdots z_n)$ の確率要素が (6) である。　よって座標軸を適当に回転して

$$X=(\xi_1, 0, 0, \cdots 0).\quad Y=(\eta_1, \eta_2, 0\cdots 0)$$

$$Z=(\zeta_1, \zeta_2, z_3'', z_4''\cdots z_n'')$$

ならしめる。　そうすれば

$$m_{11}=\xi_1^2.\quad m_{12}=\xi_1\eta_1,\quad m_{13}=\xi_1\zeta_1$$

$$m_{22}=\eta_1^2+\eta_2^2,\quad m_{23}=\eta_1\zeta_1+\eta_2\zeta_2$$

$$m_{33}=\zeta_1^2+\zeta_2^2+z_0''^2+\cdots+z_n''^2$$

これは直交変換だから，$\zeta_1, \zeta_2, Z_3'', \cdots\cdots, Z_n''$ の分布は（6）から

$$\left(\frac{A_{33}}{2\pi A}\right)^{\frac{n}{2}} exp\left[-\frac{1}{2}\left\{\left(\frac{A_{11}}{A}-\frac{A_{33.11}}{A_{33}}\right)\xi_1^2+2\left(\frac{A_{12}}{A}-\frac{A_{33.12}}{A_{33}}\right)\xi_1\eta_1+\left(\frac{A_{22}}{A}-\frac{A_{33.22}}{A_{33}}\right)(\eta_1^2+\eta_2^2)\right]\right.$$

$$exp\left[-\frac{1}{2}\left(\frac{A_{33}}{A}(\zeta_1^2+\zeta_2^2+Z_3''^2+\cdots+Z_n''^2)+2\frac{A_{31}}{A}\xi_1\zeta_1+2\frac{A_{32}}{A}(\eta_1\zeta_1+\eta_2\zeta_2)\right)\right]$$

$$d\zeta_1 d\zeta_2 dZ_3'' \cdots dZ_n'' \qquad (7)$$

となつて $\zeta_1, \zeta_2, Z_2'', \cdots, Z_n''$ は互に独立である。

$$\zeta_3^2 = Z_3''^2 + \cdots\cdots + Z_n''^2$$

として $\zeta_1, \zeta_2, \zeta_3$ の同時分布を求めると

$$\left(\frac{A_{33}}{2\pi A}\right)^{3/2} exp\left[-\frac{1}{2}\left\{\left(\frac{A_{11}}{A}-\frac{A_{33.11}}{A_{33}}\right)\xi_1^2+2\left(\frac{A_{12}}{A}-\frac{A_{33.12}}{A_{33}}\right)\xi_1\eta_1+\left(\frac{A_{22}}{A}-\frac{A_{33.22}}{A_{33}}\right)(\eta_1^2\to\eta_2^2)\right\}\right]$$

$$exp\left[-\frac{1}{2}\left\{\frac{A_{33}}{A}\zeta_1^2+2\frac{A_{31}}{A}\xi_1\zeta_1+2\frac{A_{32}}{A}\eta_1\zeta_1\right\}\right]d\zeta_1\, exp\left[-\frac{1}{2}\left\{\frac{A_{33}}{A}\zeta_2^2+2\frac{A_{32}}{A}\eta_2\zeta_2\right\}\right]d\zeta_2$$

$$\frac{2}{\left(2\cdot\frac{A}{A_{33}}\right)^{\frac{n-2}{2}}\Gamma\left(\frac{n-2}{2}\right)}\, e^{-\frac{A_{33}}{2A}\zeta_3^2}\left(\zeta_3^2\right)^{-\frac{1}{2}(n-2)-\frac{1}{2}} d\zeta_3 \qquad (8)$$

同様にして X を固定したときの η_1, η_2 の分布は

$$\left(\frac{A_{33.22}}{2\pi A_{33}}\right)^{1/2} exp\left[-\frac{1}{2}\left(\frac{A_{33.11}}{A_{33}}-\frac{1}{A_{33.22}}\right)\xi_1^2\right]$$

$$\left(\frac{A_{33.22}}{2\pi A_{33}}\right)^{\frac{n-1}{2}} exp\left[-\frac{1}{2}\left(\frac{A_{33.22}}{A_{33}}(\eta_1^2+y_2'^2+\cdots+y_n'^2)+2\frac{A_{33.22}}{A_{33}}\xi_1\eta_1\right]d\eta_1 dy_2'\cdots dy_n' \qquad (9)$$

よつて η_1, η_2 の分布は，

$$\left(\frac{\Lambda_{33.22}}{2\pi\Lambda_{33}}\right)^{1/2}\exp\left[-\frac{1}{2}\left(\frac{\Lambda_{33.11}}{\Lambda_{33}}-\frac{1}{\Lambda_{33.22}}\right)\xi_1^2\right]\exp\left[-\frac{1}{2}\left(\frac{\Lambda_{33.22}}{\Lambda_{33}}\eta_1^2+2\frac{\Lambda_{33.22}}{\Lambda_{33}}\xi_1\eta_1\right)\right]d\eta_1$$

$$\frac{2}{\left(2\frac{\Lambda_{33}}{\Lambda_{33.22}}\right)^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\ e^{-\frac{\Lambda_{33.22}}{2\Lambda_{33}}\eta_2^2}(\eta_2^2)^{\frac{1}{2}(n-1)-\frac{1}{2}}d\eta_2 \qquad (10)$$

ξ_1 の分布は

$$\frac{2}{(2\Lambda_{33.22})^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\ e^{-\frac{\xi_1^2}{2\Lambda_{33.22}}}(\xi_1^2)^{\frac{1}{2}n-\frac{1}{2}n}d\xi_1 \qquad (11)$$

(8), (10), (11) をかけて $\xi_1, \eta_1, \eta_2, \zeta_1, \zeta_2, \zeta_3$ の同時分布は次のようになる。

$$\frac{8}{2^{\frac{3n}{2}}\pi^{\frac{3}{2}}\Lambda^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}\exp\left[-\frac{1}{2}\left(\frac{\Lambda_{11}}{\Lambda}\xi_1^2+\frac{\Lambda_{22}}{\Lambda}(\eta_1^2+\eta_2^2)+\frac{\Lambda_{33}}{\Lambda}(\zeta_1^2+\zeta_2^2+\zeta_3^2)\right.\right.$$

$$\left.\left.+2\frac{\Lambda_{12}}{\Lambda}\xi_1\eta_1+2\frac{\Lambda_{33}}{\Lambda}\xi_1\zeta_1+2\frac{\Lambda_{23}}{\Lambda}(\eta_1\zeta_1+\eta_2\zeta_2)\right)\right]$$

$$\xi_1^{n-1}\eta_2^{n-2}\zeta_3^{n-3}d\xi_1 d\eta_1 d\eta_2 d\zeta_1 d\zeta_2 d\zeta_3 \qquad (12)$$

ところで

$$8\xi_1^3\eta_2^2\zeta_3\, d\xi_1 \cdots d\zeta_3 = dm_{11}\cdots dm_{33} \qquad (13)$$

$$M=(\xi_1\eta_2\zeta_3)^2 \qquad (14)$$

だから M の分布は

$$\frac{\frac{n}{2}}{2^{\frac{3n}{2}}\pi^{\frac{3}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}M^{\frac{n-4}{2}}e^{-\frac{1}{2}\Phi\cdot M}dM \qquad (15)$$

となる。 (1952.9.8)