⑦ DISTRIBUTION FREEの場合の相関係数の検定について

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各要素に x, y, z の3つの標識の対応している無限母集団から n 個の標本をとった時，x, y の相関係数 r_{xy} と y, z の相関係数 r_{yz} の差 $r_{xy}-r_{yz}$ の Expectation と Variance を求めてみる。

この時

$$E(x)=E(y)=E(z)=0 \quad \text{としておく。}$$

$$P_{xy}=\frac{\Sigma xy}{n}-\bar{x}\bar{y} \qquad E(P_{xy})=\pi_{xy}$$

$$P_{yz}=\frac{\Sigma yz}{n}-\bar{y}\bar{z} \qquad E(P_{yz})=\pi_{yz}$$

$$v_x=\frac{\Sigma x^2}{n}-\bar{x}^2 \qquad E(v_x)=\varphi_x$$

$$v_y=\frac{\Sigma y^2}{n}-\bar{y}^2 \qquad E(v_y)=\varphi_y$$

$$v_z=\frac{\Sigma z^2}{n}-\bar{z}^2 \qquad E(v_z)=\varphi_z$$

とする。

$$\pi_{xy}=E(P_{xy})=E\left(\frac{\Sigma x_i y_i}{n}-\bar{x}\bar{y}\right)=E(xy)-E(\bar{x}\bar{y})$$

$$=E(xy)-\frac{1}{n^2}E\{(x_1+\cdots+x_n)(y_1+\cdots+y_n)\}=E(xy)-\frac{1}{n^2}\{nE(x_iy_i)+n(n-1)E(x_iy_i)\}$$

$$= \frac{n-1}{n} E(xy) - \frac{n-1}{n} E(x) E(y) = \frac{n-1}{n} E(xy)$$

$$\varphi_x = E(v_x) = E\left(\frac{\Sigma x^2}{n} - \bar{x}^2 \right) = E(x^2) - E(\bar{x}^2)$$

$$= E(x^2) - \frac{1}{n^2} \left\{ n E(x^2) - n(n-1) E(x) E(x) \right\}$$

$$= \frac{n-1}{n} E(x^2)$$

$$E\left\{ \frac{(v_x - \varphi_x)^2}{\varphi_x^2} \right\} = \frac{E(v_x^2)}{\varphi_x^2} - 1$$

$$= \frac{1}{\varphi_x^2} E\left\{ \left(\frac{\Sigma x^2}{n} - \bar{x}^2 \right)^2 \right\} - 1$$

$$= \frac{1}{\varphi_x^2} \left\{ \frac{(\Sigma x^2)^2}{n^2} - 2\bar{x}^2 \frac{\Sigma x^2}{n} + \bar{x}^4 \right\} - 1$$

$$= \frac{1}{\varphi_x^2} \left\{ \frac{1}{n} E(x^4) + \frac{n-1}{n} \{E(x^2)\}^2 - \frac{2}{n^2} E(x^4) - \frac{2(n-1)}{n^2} \{E(x^2)\}^2 \right.$$

$$\left. + \frac{1}{n^3} E(x^4) + \frac{3(n-1)}{n^3} \{E(x^2)\}^2 \right\} - 1$$

$$= \frac{1}{\left(\frac{n-1}{n}\right)^2 \{E(x^2)\}^2} \cdot \left\{ \frac{(n-1)^2}{n^3} E(x^4) + \frac{(n-1)(n^2-2n+3)}{n^3} \{E(x^2)\}^2 \right\} - 1$$

$$= \frac{1}{n} \cdot \frac{E(x^4)}{\{E(x^2)\}^2} - \frac{n-3}{n(n-1)}$$

$$E\left\{ \frac{(P_{xy} - \pi_{xy})(v_x - \varphi_x)}{\pi_{xy} \varphi_x} \right\} = \frac{E(P_{xy} v_x)}{\pi_{xy} \varphi_x} - 1$$

$$= \frac{E\left\{ \left(\frac{\Sigma xy}{n} - \bar{x}\bar{y} \right) \left(\frac{\Sigma x^2}{n} - \bar{x}^2 \right) \right\}}{\pi_{xy} \varphi_x} - 1$$

$$= \frac{1}{\pi_{xy} \varphi_x} E\left\{ \frac{\Sigma xy \Sigma x^2}{n^2} - \bar{x}\bar{y} \frac{\Sigma x^2}{n} - \frac{\Sigma xy}{n} \bar{x}^2 + \bar{x}^3 \bar{y} \right\} - 1$$

$$= \frac{1}{\pi_{xy}\varphi_x}\left\{ \frac{E(x^3y)}{n} + \frac{n-1}{n}E(x^2)E(xy) - \frac{E(x^3y)}{n^2} - \frac{n-1}{n^2}E(x^2)E(xy) \right.$$

$$\left. - \frac{E(x^3y)}{n} - \frac{n-1}{n^2}E(x^2)E(xy) + \frac{E(x^3y)}{n^3} + \frac{3(n-1)}{n^3}E(x^2)E(xy)\right\} - 1$$

$$= \frac{1}{\left(\frac{n-1}{n}\right)^2 E(xy)E(x^2)}\left\{ \frac{(n-1)^2}{n^3}E(x^3y) + \frac{(n-1)(n^2-2n+3)}{n^3}E(x^2)E(xy)\right\} - 1$$

$$= \frac{1}{n}\cdot\frac{E(x^3y)}{E(x^2)E(xy)} - \frac{n-3}{n(n-1)}$$

$$E\left\{\frac{(v_x-\varphi_x)(v_y-\varphi_y)}{\varphi_x\varphi_y}\right\} = \frac{E(v_x v_y)}{\varphi_x\varphi_y} - 1$$

$$= \frac{1}{\varphi_x\varphi_y}E\left\{\left(\frac{\Sigma x^2}{n} - \bar{x}^2\right)\left(\frac{\Sigma y^2}{n} - \bar{y}^2\right)\right\} - 1$$

$$= \frac{1}{\varphi_x\varphi_y}\cdot E\left(\frac{\Sigma x^2 \Sigma y^2}{n^2} - \bar{x}^2\frac{\Sigma y^2}{n} - \bar{y}^2\frac{\Sigma x^2}{n} + \bar{x}^2\bar{y}^2\right) - 1$$

$$= \frac{1}{\varphi_x\varphi_y}\cdot\left(\frac{E(x^2y^2)}{n} + \frac{n-1}{n}E(x^2)E(y^2) - \frac{E(x^2y^2)}{n^2} - \frac{n-1}{n^2}E(x^2)E(y^2) \right.$$

$$\left. - \frac{E(x^2y^2)}{n^2} - \frac{n-1}{n^2}E(x^2)E(y^2) + \frac{E(x^2y^2)}{n^3} + \frac{3(n-1)}{n^3}E(x^2)E(y^2)\right\} - 1$$

$$= \frac{1}{\left(\frac{n-1}{n}\right)^2\cdot E(x^2)E(y^2)}\cdot\left\{ \frac{(n-1)^2}{n^3}\cdot E(x^2y^2) + \frac{(n-1)(n^2-2n+3)}{n^3}E(x^2)E(y^2)\right\} - 1$$

$$= \frac{1}{n}\cdot\frac{E(x^2y^2)}{E(x^2)E(y^2)} - \frac{n-3}{n(n-1)}$$

$$E\left\{\frac{(P_{xy}-\pi_{xy})(P_{yz}-\pi_{yz})}{\pi_{xy}\pi_{yz}}\right\} = \frac{E(P_{xy}P_{yz})}{\pi_{xy}\pi_{yz}} - 1$$

$$= \frac{1}{\pi_{xy}\pi_{yz}}\left[E\left\{\left(\frac{\Sigma xy}{n} - \bar{x}\bar{y}\right)\left(\frac{\Sigma yz}{n} - \bar{y}\bar{z}\right)\right\}\right] - 1$$

$$= \frac{1}{\pi_{xy}\pi_{yz}} E\left\{ \frac{\Sigma xy}{n} \cdot \frac{\Sigma yz}{n} - \bar{x}\bar{y}\frac{\Sigma yz}{n} - \bar{y}\bar{z}\frac{\Sigma xy}{n} + \bar{x}\bar{y}^2\bar{z} \right\}$$

$$= \frac{1}{\pi_{xy}\pi_{yz}} \left\{ \frac{E(xy^2z)}{n} + \frac{n-1}{n}E(xy)E(yz) - \frac{E(xy^2z)}{n^2} - \frac{n-1}{n^2}E(xy)E(yz) \right.$$

$$- \frac{1}{n^2}E(xy^2z) - \frac{n-1}{n^2}E(xy)E(yz) + \frac{1}{n^3}E(xy^2z)$$

$$\left. + \frac{n-1}{n^3}E(y^2)E(xz) + \frac{2(n-1)}{n^3}E(xy)E(yz) \right\} - 1.$$

$$= \frac{1}{(\frac{n-1}{n})^2 \cdot E(xy)E(yz)} \left\{ \frac{(n-1)^2}{n^3}E(xy^2z) + \frac{(n-1)(n^2-2n+2)}{n^3}E(xy)E(yz) \right.$$

$$\left. + \frac{n-1}{n^3}E(y^2)E(xz) \right\} - 1$$

$$= \frac{1}{n}\frac{E(xy^2z)}{E(xy)E(yz)} - \frac{n-2}{n(n-1)} + \frac{1}{n(n-1)}\frac{E(y^2)E(xz)}{E(xy)E(yz)}$$

$$E\left\{ \frac{(P_{xy}-\pi_{xy})(\upsilon_z-\varphi_z)}{\pi_{xy}\varphi_z} \right\} = \frac{E(P_{xy}\upsilon_z)}{\pi_{xy}\varphi_z} - 1$$

$$= \frac{1}{\pi_{xy}\varphi_z} \cdot E\left\{ \left(\frac{\Sigma xy}{n} - \bar{x}\bar{y}\right)\left(\frac{\Sigma z^2}{n} - \bar{z}^2\right) \right\} - 1$$

$$= \frac{1}{\pi_{xy}\varphi_z} \cdot E\left\{ \frac{\Sigma xy}{n}\frac{\Sigma z^2}{n} - \bar{x}\bar{y}\frac{\Sigma z^2}{n} - \bar{z}^2\frac{\Sigma xy}{n} + \bar{x}\bar{y}\bar{z}^2 \right\} - 1$$

$$= \frac{1}{\pi_{xy} \cdot \varphi_z} \left\{ \frac{E(xyz^2)}{n} + \frac{n-1}{n}E(xy)E(z^2) - \frac{E(xyz^2)}{n^2} \right.$$

$$- \frac{n-1}{n^2}E(xy)E(z^2) - \frac{E(xyz^2)}{n^2} - \frac{n-1}{n^2}E(xy)E(z^2)$$

$$\left. + \frac{E(xyz^2)}{n^3} + \frac{n-1}{n^3}E(xy)E(z^2) + \frac{2(n-1)}{n^3}E(xz)E(yz) \right\} - 1$$

$$= \frac{1}{(\frac{n-1}{n})^2 E(xy)E(z^2)} \left\{ \frac{(n-1)^2}{n^3}E(xyz^2) + \frac{(n-1)^3}{n^3}E(xy)E(z^2) + \frac{2(n-1)}{n^3}E(xz)E(yz) \right\}$$

$$-1$$

$$= \frac{1}{n} \cdot \frac{E(xyz^2)}{E(xy)E(z^2)} + \frac{2}{n(n-1)} \frac{E(xz)E(yz)}{E(xy)E(z^2)} - \frac{1}{n}$$

$$\frac{E\{(P_{xy} - \pi_{xy})^2\}}{\pi^2_{xy}} = \frac{E(P^2_{xy}) - \pi^2_{xy}}{\pi^2_{xy}} = \frac{E(P^2_{xy})}{\pi^2_{xy}} - 1$$

$$= \frac{1}{\pi^2_{xy}} \cdot E\left\{\left(\frac{\Sigma xy}{n} - \bar{x}\bar{y}\right)^2\right\} - 1$$

$$= \frac{1}{\pi^2_{xy}} \cdot E\left(\frac{(\Sigma xy)^2}{n^2} - 2\bar{x}\bar{y}\frac{\Sigma xy}{n} + \bar{x}^2\bar{y}^2\right) - 1$$

$$= \frac{1}{\pi^2_{xy}} \cdot \left\{ \frac{E(x^2y^2)}{n} + \frac{n-1}{n}\{E(xy)\}^2 - \frac{2}{n^3}\left[nE(x^2y^2) + n(n-1)\{E(xy)\}^2\right]\right.$$

$$\left. + \frac{1}{n^4}\left[nE(x^2y^2) + 2n(n-1)\{E(xy)\}^2 + n(n-1)E(x^2)E(y^2)\right]\right\} - 1$$

$$= \frac{1}{\left(\frac{n-1}{n}\right)^2\{E(xy)\}^2}\left[\frac{E(x^2y^2)}{n} + \frac{n-1}{n}\{E(xy)\}^2 - \frac{2}{n^2}E(x^2y^2) - \frac{2(n-1)}{n^2}\{E(xy)\}^2\right.$$

$$\left. + \frac{E(x^2y^2)}{n^3} + \frac{2(n-1)}{n^3}\{E(xy)\}^2 + \frac{n-1}{n^3}E(x^2)E(y^2)\right] - 1$$

$$= \frac{1}{\left(\frac{n-1}{n}\right)^2\{E(xy)\}^2}\left[\frac{(n-1)^2}{n^3}E(x^2y^2) + \frac{(n-1)(n^2-2n+2)}{n^3}\{E(xy)\}^2 + \frac{n-1}{n^3}E(x^2)E(y^2)\right] - 1$$

$$= \frac{1}{\left(\frac{n-1}{n}\right)^2\{E(xy)\}^2}\left[\frac{(n-1)^2}{n^3}E(x^2y^2) + \frac{(n-1)(n^2-2n-2)}{n^3}\{E(xy)\}^2 + \frac{n-1}{n^3}E(x^2)E(y^2)\right] - 1$$

$$= \frac{1}{n} \cdot \frac{E(x^2y^2)}{\{E(xy)\}^2} + \frac{(n^2-2n+2)}{n(n-1)} - 1 + \frac{1}{n(n-1)} \frac{E(x^2)E(y^2)}{\{E(xy)\}^2}$$

$$= \frac{1}{n} \frac{E(x^2y^2)}{\{E(xy)\}^2} - \frac{n-2}{n(n-1)} + \frac{1}{n(n-1)} \frac{E(x^2)E(y^2)}{\{E(xy)\}^2}$$

$$E(r_{xy}) = E\left(\frac{\frac{\Sigma xy}{n} - \bar{x}\bar{y}}{\sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}\sqrt{\frac{\Sigma y^2}{n} - \bar{y}^2}}\right) = E\left(\frac{P_{xy}}{\sqrt{v_x}\sqrt{v_y}}\right)$$

$$= E\left\{\frac{\pi_{xy} + P_{xy} - \pi_{xy}}{(\varphi_x + v_x - \varphi_x)^{\frac{1}{2}}(\varphi_y + v_y - \varphi_y)^{\frac{1}{2}}}\right\}$$

$$= \frac{\pi_{xy}}{\varphi_x^{\frac{1}{2}}\varphi_y^{\frac{1}{2}}} E\left\{\left(1 + \frac{P_{xy} - \pi_{xy}}{\pi_{xy}}\right)\left(1 + \frac{v_x - \varphi_x}{\varphi_x}\right)^{-\frac{1}{2}}\left(1 + \frac{v_y - \varphi_y}{\varphi_y}\right)^{-\frac{1}{2}}\right\}$$

$$= \frac{E(xy)}{\sqrt{E(x^2)}\sqrt{E(y^2)}} E\left[\left(1 + \frac{P_{xy} - \pi_{xy}}{\pi_{xy}}\right)\left\{1 - \frac{1}{2}\cdot\frac{v_x - \varphi_x}{\varphi_x} + \frac{3}{8}\left(\frac{v_x - \varphi_x}{\varphi_x}\right)^2 + \cdots\right\}\right.$$

$$\left.\left\{1 - \frac{1}{2}\frac{v_y - \varphi_y}{\varphi_y} + \frac{3}{8}\left(\frac{v_y - \varphi_y}{\varphi_y}\right)^2 + \cdots\right\}\right]$$

$$= \rho_{xy}\left\{1 - \frac{1}{2}\cdot\frac{E\{(P_{xy} - \pi_{xy})(v_x - \varphi_x)\}}{\pi_{xy}\varphi_x} - \frac{1}{2}\frac{E\{(P_{xy} - \pi_{xy})(v_y - \varphi_y)\}}{\pi_{xy}\varphi_y}\right.$$

$$\left. + \frac{1}{4}\cdot\frac{E\{(v_x - \varphi_x)(v_y - \varphi_y)\}}{\varphi_x\varphi_y} + \frac{3}{8}\frac{E\{(v_x - \varphi_x)^2\}}{\varphi_x^2} + \frac{3}{8}\frac{E\{(v_y - \varphi_y)^2\}}{\varphi_y^2} + \cdots\right]$$

$$= \rho_{xy}\left\{1 - \frac{1}{2}\left(\frac{1}{n}\cdot\frac{E(x^3y)}{E(x^2)E(xy)} - \frac{n-3}{n(n-1)}\right) - \frac{1}{2}\left(\frac{1}{n}\frac{E(xy^3)}{E(y^2)E(xy)} - \frac{n-3}{n(n-1)}\right)\right.$$

$$+ \frac{1}{4}\left(\frac{1}{n}\cdot\frac{E(x^2y^2)}{E(x^2)E(y^2)} - \frac{n-3}{n(n-1)}\right) + \frac{3}{8}\left(\frac{1}{n}\cdot\frac{E(x^4)}{\{E(x^2)\}^2} - \frac{n-3}{n(n-1)}\right\}$$

$$\left. + \frac{3}{8}\left(\frac{1}{n}\cdot\frac{E(y^4)}{\{E(y^2)\}^2} - \frac{n-3}{n(n-1)}\right) + O\left(\frac{1}{n^2}\right)\right\}$$

$$= \rho_{xy}\left\{1 - \frac{1}{n}\left(\frac{1}{2}\cdot\frac{E(x^3y)}{E(x^2)E(xy)} + \frac{1}{2}\frac{E(xy^3)}{E(y^2)E(xy)} - \frac{1}{4}\cdot\frac{E(x^2y^2)}{E(x^2)E(y^2)}\right.\right.$$

$$\left.\left. - \frac{3}{8}\cdot\frac{E(x^4)}{\{E(x^2)\}^2} - \frac{3}{8}\frac{E(y^4)}{\{E(y^2)\}^2} + O\left(\frac{1}{n^2}\right)\right\}\right.$$

$E(r_{yz})$ も同様.実用的には $\frac{1}{n}$ の order 以下をはぶいて $E(r_{xy}) = \rho_{xy}$ としてもよい.

$$E(r^2_{xy}) = E\left(\frac{p^2_{xy}}{v_x v_y}\right) = \frac{\pi^2_{xy}}{\varphi_x \varphi_y} E\left\{\left(1+\frac{p_{xy}-\pi_{xy}}{\pi_{xy}}\right)^2\left(1+\frac{v_x-\varphi_x}{\varphi_x}\right)^{-1} \times\left(1+\frac{v_y-\varphi_y}{\varphi_x}\right)^{-1}\right\}$$

$$= \rho^2 \cdot E\left\{\left(1+2\cdot\frac{p_{xy}-\pi_{xy}}{\pi_{xy}}+\frac{(p_{xy}-\pi_{xy})^2}{\pi^2_{xy}}\right)\left(1-\frac{v_x-\varphi_x}{\varphi_x}+\frac{(v_x-\varphi_x)^2}{\varphi^2_x}-\cdots\right)\left(1-\frac{v_y-\varphi_y}{\varphi_y}+\frac{(v_y-\varphi_y)^2}{\varphi^2_y}+\cdots\cdots\right)\right\}$$

$$= \rho^2\left[1-2\frac{E\{(p_{xy}-\pi_{xy})(v_x-\varphi_x)\}}{\pi_{xy}\varphi_x}-2\frac{E\{(p_{xy}-\pi_{xy})(v_y-\varphi_y)\}}{\pi_{xy}\varphi_y}+\frac{E\{(p_{xy}-\pi_{xy})^2\}}{\pi^2_{xy}}+\frac{E\{(v_x-\varphi_x)^2\}}{\varphi^2_x}+\frac{E\{(v_y-\varphi_y)^2\}}{\varphi^2_y}+\frac{E\{(v_x-\varphi_x)(v_y-\varphi_y)\}}{\varphi_x\varphi_y}+\cdots\right]$$

$$= \rho^2\left[1-\frac{2}{n}\cdot\frac{E(x^3y)}{E(x^2)E(xy)}+\frac{2(n-3)}{n(n-1)}-\frac{2}{n}\frac{E(xy^3)}{E(y^2)E(xy)}+\frac{2(n-3)}{n(n-1)}\right.$$
$$+\frac{1}{n}\cdot\frac{E(x^2y^2)}{\{E(xy)\}^2}-\frac{n-2}{n(n-1)}+\frac{1}{n(n-1)}\frac{E(x^2)E(y^2)}{\{E(xy)\}^2}$$
$$+\frac{1}{n}\frac{E(x^4)}{\{E(x^2)\}^2}-\frac{n-3}{n(n-1)}+\frac{1}{n}\cdot\frac{E(y^4)}{\{E(y^2)\}^2}-\frac{n-3}{n(n-1)}$$
$$\left.+\frac{1}{n}\frac{E(x^2y^2)}{E(x^2)E(y^2)}-\frac{n-3}{n(n-1)}+\cdots\cdots O\left(\frac{1}{n^2}\right)\right]$$

$$= \rho^2\left[1-\frac{2}{n}\frac{E(x^3y)}{E(x^2)E(xy)}-\frac{2}{n}\frac{E(xy^3)}{E(y^2)E(xy)}+\frac{1}{n}\cdot\frac{E(x^2y^2)}{\{E(xy)\}^2}\right.$$
$$\left.+\frac{1}{n}\frac{E(x^4)}{\{E(x^2)\}^2}+\frac{1}{n}\frac{E(y^4)}{\{E(y^2)\}^2}+\frac{1}{n}\frac{E(x^2y^2)}{E(x^2)E(y^2)}+O\left(\frac{1}{n^2}\right)\right]$$

$$E(r_{xy}^2)-\{E(r_{xy})\}^2=\rho^2\Big[1-\frac{2}{n}\frac{E(x^3y)}{E(x^2)E(xy)}-\frac{2}{n}\cdot\frac{E(xy^3)}{E(y^2)E(xy)}$$

$$+\frac{1}{n}\frac{E(x^2y^2)}{\{E(xy)\}^2}+\frac{1}{n}\frac{E(x^4)}{\{E(x^2)\}^2}+\frac{1}{n}\frac{E(y^4)}{\{E(y^2)\}^2}+\frac{1}{n}\frac{E(x^2y^2)}{E(x^2)E(y^2)}+\cdots O\left(\frac{1}{n^2}\right)\Big]$$

$$-\rho^2\Big[1-\frac{1}{n}\cdot\frac{1}{2}\cdot\frac{E(x^3y)}{E(x^2)E(xy)}-\frac{1}{n}\cdot\frac{1}{2}\cdot\frac{E(xy^3)}{E(y^2)E(xy)}+\frac{1}{n}\cdot\frac{1}{4}\cdot\frac{E(x^2y^2)}{E(x^2)E(y^2)}$$

$$+\frac{1}{n}\cdot\frac{3}{8}\cdot\frac{E(x^4)}{\{E(x^2)\}^2}+\frac{1}{n}\cdot\frac{3}{8}\cdot\frac{E(y^4)}{\{E(y^2)\}^2}+\cdots O\left(\frac{1}{n^2}\right)\Big]^2$$

$$=\rho^2\Big[-\frac{1}{n}\cdot\frac{E(x^3y)}{E(x^2)E(xy)}-\frac{1}{n}\cdot\frac{E(xy^3)}{E(y^2)E(xy)}+\frac{1}{n}\cdot\frac{E(x^2y^2)}{\{E(xy)\}^2}$$

$$+\frac{1}{4n}\frac{E(x^4)}{\{E(x^2)\}^2}+\frac{1}{4n}\frac{E(y^4)}{\{E(y^2)\}^2}+\frac{1}{2n}\cdot\frac{E(x^2y^2)}{E(x^2)E(y^2)}$$

$$O\left(\frac{1}{n^2}\right)\Big]$$

$$E(r_{xy}r_{yz})=E\left(\frac{P_{xy}\,P_{yz}}{\sqrt{v_x\cdot v_y\cdot v_y\cdot v_z}}\right)$$

$$=E\left\{\frac{\pi_{xy}\pi_{yz}}{\sqrt{\varphi_x\varphi_y\varphi_y\varphi_z}}\left(1+\frac{P_{xy}-\pi_{xy}}{\pi_{xy}}\right)\left(1+\frac{P_{yz}-\pi_{yz}}{\pi_{yz}}\right)\left(1+\frac{v_x-\varphi_x}{\varphi_x}\right)^{-\frac{1}{2}}\left(1+\frac{v_y-\varphi_y}{\varphi_y}\right)^{-\frac{1}{2}}\right.$$

$$\left.\times\left(1+\frac{v_y-\varphi_y}{\varphi_y}\right)^{-\frac{1}{2}}\left(1+\frac{v_z-\varphi_z}{\varphi_z}\right)^{-\frac{1}{2}}\right\}$$

$$=\rho_{xy}\rho_{yz}\cdot E\left\{\left(1+\frac{P_{xy}-\pi_{xy}}{\pi_{xy}}\right)\left(1+\frac{P_{yz}-\pi_{yz}}{\pi_{yz}}\right)\left(1+\frac{v_x-\varphi_x}{\varphi_x}\right)^{-\frac{1}{2}}\left(1+\frac{v_y-\varphi_y}{\varphi_y}\right)^{-1}\left(1+\frac{v_z-\varphi_z}{\varphi_z}\right)^{-\frac{1}{2}}\right\}$$

$$=\rho_{xy}\rho_{yz}\cdot E\Big\{\left(1+\frac{P_{xy}-\pi_{xy}}{\pi_{xy}}\right)\left(1+\frac{P_{yz}-\pi_{yz}}{\pi_{yz}}\right)\left(1-\frac{1}{2}\cdot\frac{v_x-\varphi_x}{\varphi_x}+\frac{3}{8}\frac{(v_x-\varphi_x)^2}{\varphi_x^2}+\cdots\right)$$

$$\left(1-\frac{v_y-\varphi_y}{\varphi_y}+\frac{(v_y-\varphi_y)^2}{\varphi_y^2}+\cdots\right)\left(1-\frac{1}{2}\frac{(v_z-\varphi_z)}{\varphi_z}+\frac{3}{8}\frac{(v_z-\varphi_z)^2}{\varphi_z^2}+\cdots\right)\Big\}$$

$$=\rho_{xy}\rho_{yz}\Big[1+\frac{E\{(P_{xy}-\pi_{xy})(P_{yz}-\pi_{yz})\}}{\pi_{xy}\pi_{yz}}-\frac{1}{2}\frac{E\{(P_{xy}-\pi_{xy})(v_x-\varphi_x)\}}{\pi_{xy}\varphi_x}$$

$$- \frac{E\{(P_{xy}-\pi_{xy})(\nu_y-\varphi_y)\}}{\pi_{xy}\varphi_y} - \frac{1}{2}\frac{E\{(P_{xy}-\pi_{xy})(\nu_z-\varphi_z)\}}{\pi_{xy}\varphi_z}$$

$$- \frac{1}{2}\frac{E\{(P_{yz}-\pi_{yz})(\nu_x-\varphi_x)\}}{\pi_{yz}\varphi_x} - \frac{E\{(P_{yz}-\pi_{yz})(\nu_y-\varphi_y)\}}{\pi_{yz}\varphi_y}$$

$$- \frac{1}{2}\frac{E\{(P_{yz}-\pi_{yz})(\nu_z-\varphi_z)\}}{\pi_{yz}\varphi_z} + \frac{1}{2}\frac{E\{(\nu_x-\varphi_x)(\nu_y-\varphi_y)\}}{\varphi_x\varphi_y}$$

$$+ \frac{1}{2}\frac{E\{(\nu_y-\varphi_y)(\nu_z-\varphi_z)\}}{\varphi_y\varphi_z} + \frac{1}{4}\cdot\frac{E\{(\nu_x-\varphi_x)(\nu_z-\varphi_z)\}}{\varphi_x\varphi_z}$$

$$+ \frac{3}{8}\frac{E\{(\nu_x-\varphi_x)^2\}}{\varphi_x^2} + \frac{E\{(\nu_y-\varphi_y)^2\}}{\varphi_y^2} + \frac{3}{8}\frac{E\{(\nu_z-\varphi_z)^2\}}{\varphi_z^2}$$

$$+ \cdots \Big]$$

$$= \rho_{xy}\rho_{yz}\Big[1 + \frac{1}{n}\frac{E(xy^2z)}{E(xy)E(yz)} - \frac{n-2}{n(n-1)} + \frac{1}{n(n-1)}\frac{E(y^2)E(xz)}{E(xy)E(yz)}$$

$$- \frac{1}{2}\frac{1}{n}\frac{E(x^3y)}{E(x^2)E(xy)} + \frac{1}{2}\cdot\frac{n-3}{n(n-1)}$$

$$- \frac{1}{n}\frac{E(xy^3)}{E(y^2)E(xy)} + \frac{n-3}{n(n-1)}$$

$$- \frac{1}{2}\cdot\frac{1}{n}\frac{E(xyz^2)}{E(z^2)E(xy)} + \frac{1}{2}\frac{1}{n} - \frac{1}{n(n-1)}\frac{E(xz)E(yz)}{E(z^2)E(xy)}$$

$$- \frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(x^2yz)}{E(x^2)E(yz)} + \frac{1}{2}\cdot\frac{1}{n} - \frac{1}{n(n-1)}\frac{E(xy)E(xz)}{E(x^2)E(yz)}$$

$$- \frac{1}{n}\frac{E(y^3z)}{E(y^2)E(yz)} + \frac{n-3}{n(n-1)}$$

$$- \frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(yz^3)}{E(z^2)E(yz)} + \frac{1}{2}\frac{n-3}{n(n-1)}$$

$$+ \frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(x^2y^2)}{E(x^2)E(y^2)} - \frac{1}{2}\cdot\frac{n-3}{n(n-1)}$$

$$+\frac{1}{2}\frac{1}{n}\frac{E(y^2z^2)}{E(y^2)E(z^2)}-\frac{1}{2}\frac{(n-3)}{n(n-1)}$$

$$+\frac{1}{4}\frac{1}{n}\frac{E(x^2z^2)}{E(x^2)E(z^2)}-\frac{1}{4}\frac{(n-3)}{n(n-1)}$$

$$+\frac{3}{8}\cdot\frac{1}{n}\frac{E(x^4)}{\{E(x^2)\}^2}-\frac{3}{8}\cdot\frac{(n-3)}{n(n-1)}$$

$$+\frac{1}{n}\frac{E(y^4)}{\{E(y^2)\}^2}-\frac{(n-3)}{n(n-1)}$$

$$+\frac{3}{8}\cdot\frac{1}{n}\frac{E(z^4)}{\{E(z^2)\}^2}-\frac{3}{8}\frac{n-3}{n(n-1)}$$

$$+\cdot O\left(\frac{1}{n^2}\right) \quad \Big]$$

$$E(r_{xy}r_{yz})-E(r_{xy})E(r_{yz})$$

$$=\rho_{xy}\rho_{yz}\Big(1+\frac{1}{n}\frac{E(xy^2z)}{E(xy)E(yz)}-\frac{1}{2}\frac{1}{n}\frac{E(x^3y)}{E(x^2)E(xy)}-\frac{1}{n}\frac{E(xy^3)}{E(y^2)E(xy)}$$

$$-\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(xyz^2)}{E(z^2)E(xy)}-\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(x^2yz)}{E(x^2)E(yz)}-\frac{1}{n}\cdot\frac{E(y^3z)}{E(y^2)E(yz)}$$

$$-\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(yz^3)}{E(z^2)E(yz)}+\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(x^2y^2)}{E(x^2)E(y^2)}+\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(y^2z^2)}{E(y^2)E(z^2)}$$

$$+\frac{1}{4}\cdot\frac{1}{n}\cdot\frac{E(x^2z^2)}{E(x^2)E(z^2)}+\frac{3}{8}\cdot\frac{1}{n}\cdot\frac{E(x^4)}{\{E(x^2)\}^2}+\frac{1}{n}\cdot\frac{E(y^4)}{\{E(y^2)\}^2}$$

$$+\frac{3}{8}\cdot\frac{1}{n}\frac{E(z^4)}{\{E(z^2)\}^2}+\cdots O\left(\frac{1}{n^2}\right)\qquad\Big)$$

$$-\rho_{xy}\rho_{yz}\Big(1-\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(x^3y)}{E(x^2)E(xy)}-\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(xy^3)}{E(y^2)E(xy)}+\frac{1}{4}\cdot\frac{1}{n}\frac{E(x^2y^2)}{E(x^2)E(y^2)}$$

$$+\frac{3}{8}\cdot\frac{1}{n}\cdot\frac{E(x^4)}{\{E(x^2)\}^2}+\frac{3}{8}\cdot\frac{1}{n}\cdot\frac{E(y^4)}{\{E(y^2)\}^2}+O\left(\frac{1}{n^2}\right)\Big)\times$$

$$\Big(1-\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(y^3z)}{E(y^2)E(yz)}-\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{E(yz^3)}{E(z^2)E(yz)}+\frac{1}{4}\cdot\frac{1}{n}\cdot\frac{E(y^2z^2)}{E(y^2)E(z^2)}\cdot$$

$$+\frac{3}{8}\cdot\frac{1}{n}\frac{E(y^4)}{\{E(y^2)\}^2}+\frac{3}{8}\cdot\frac{1}{n}\cdot\frac{E(z^4)}{\{E(z^2)\}^2}+\cdots O\left(\frac{1}{n^2}\right)\Bigg)$$

$$=\rho_{xy}\rho_{yz}\times\frac{1}{n}\times\Bigg(\frac{E(xy^2z)}{E(xy)E(yz)}-\frac{1}{2}\cdot\frac{E(xy^3)}{E(y^2)E(xy)}-\frac{1}{2}\cdot\frac{E(y^3z)}{E(y^2)E(yz)}$$

$$-\frac{1}{2}\frac{E(x^2yz)}{E(x^2)E(yz)}-\frac{1}{2}\frac{E(xyz^2)}{E(z^2)E(xy)}$$

$$+\frac{1}{4}\frac{E(x^2y^2)}{E(x^2)E(y^2)}+\frac{1}{4}\frac{E(y^2z^2)}{E(y^2)E(z^2)}+\frac{1}{4}\cdot\frac{E(x^2z^2)}{E(x^2)E(z^2)}$$

$$+\frac{1}{4}\frac{E(y^4)}{\{E(y^2)\}^2}\Bigg)+O\left(\frac{1}{n^2}\right)$$

$$\sigma^2(r_{xy}-r_{yz})=E\{(r_{xy}-r_{yz})^2\}-\{E(r_{xy}-r_{yz})\}^2$$

$$=E(r_{xy}^2-2r_{xy}r_{yz}+r_{yz}^2)-\{E(r_{xy})-E(r_{yz})\}^2$$

$$=\left[E(r_{xy}^2)-\{E(r_{xy})\}^2\right]+\left[E(r_{yz}^2)-\{E(r_{yz})\}^2\right]$$

$$-2\left[E(r_{xy}r_{yz})-E(r_{xy})E(r_{yz})\right]$$

$$=\frac{1}{n}\cdot\rho_{xy}^2\Bigg(-\frac{E(x^3y)}{E(x^2)E(xy)}-\frac{E(xy^3)}{E(y^2)E(xy)}+\frac{E(x^2y^2)}{\{E(xy)\}^2}+\frac{1}{4}\cdot\frac{E(x^4)}{\{E(x^2)\}^2}$$

$$+\frac{1}{4}\cdot\frac{E(y^4)}{\{E(y^2)\}^2}+\frac{1}{2}\cdot\frac{E(x^2y^2)}{E(x^2)E(y^2)}+\cdots O\left(\frac{1}{n}\right)\Bigg)$$

$$+\frac{1}{n}\cdot\rho_{yz}^2\Bigg(-\frac{E(y^3z)}{E(y^2)E(yz)}-\frac{E(yz^3)}{E(z^2)E(yz)}+\frac{E(z^2y^2)}{\{E(zy)\}^2}+\frac{1}{4}\frac{E(y^4)}{\{E(y^2)\}^2}$$

$$+\frac{1}{4}\frac{E(z^4)}{\{E(z^2)\}^2}+\frac{1}{2}\cdot\frac{E(y^2z^2)}{E(y^2)E(z^2)}+\cdots O\left(\frac{1}{n}\right)\Bigg)$$

$$-\frac{2}{n}\cdot\rho_{xy}\rho_{yz}\Bigg(\frac{E(xy^2z)}{E(xy)E(yz)}-\frac{1}{2}\cdot\frac{E(xy^3)}{E(y^2)E(xy)}-\frac{1}{2}\cdot\frac{E(y^3z)}{E(y^2)E(yz)}$$

$$-\frac{1}{2}\cdot\frac{E(x^2yz)}{E(x^2)E(yz)}-\frac{1}{2}\cdot\frac{E(xyz^2)}{E(z^2)E(xy)}+\frac{1}{4}\frac{E(x^2y^2)}{E(x^2)E(y^2)}$$

$$+\frac{1}{4}\cdot\frac{E(y^2z^2)}{E(y^2)E(z^2)}+\frac{1}{4}\frac{E(x^2z^2)}{E(x^2)E(z^2)}+\frac{1}{4}\frac{E(y^4)}{\{E(y^2)\}^2}+o\left(\frac{1}{n}\right)\Big)$$

$$=\frac{1}{n}\cdot\Big\{\frac{1}{4}\cdot\frac{E(y^4)}{\{E(y^2)\}^2}(\rho_{xy}-\rho_{yz})^2+\frac{1}{2}\cdot\frac{E(x^2y^2)}{E(x^2)E(y^2)}(\rho_{xy}^2-\rho_{xy}\rho_{yz})$$

$$-\frac{E(xy^3)}{E(y^2)E(xy)}(\rho_{xy}^2-\rho_{xy}\rho_{yz})+\frac{1}{2}\cdot\frac{E(y^2z^2)}{E(y^2)E(z^2)}(\rho_{yz}^2-\rho_{xy}\rho_{yz})$$

$$-\frac{E(y^3z)}{E(y^2)E(yz)}(\rho_{yz}^2-\rho_{xy}\rho_{yz})+\frac{1}{4}\frac{E(x^4)}{\{E(x^2)\}^2}\rho_{xy}^2$$

$$+\frac{1}{4}\frac{E(z^4)}{\{E(z^2)\}^2}\rho_{yz}^2-\frac{1}{2}\cdot\frac{E(x^2z^2)}{E(x^2)E(z^2)}\rho_{xy}\rho_{yz}+\frac{E(x^2y^2)}{\{E(xy)\}^2}\rho_{xy}^2$$

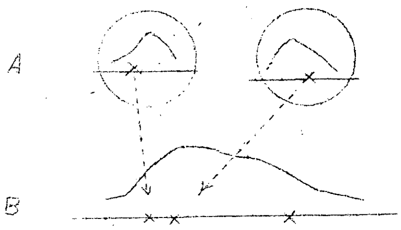
$$-\frac{E(x^3y)}{E(x^2)E(xy)}\rho_{xy}^2+\frac{E(y^2z^2)}{\{E(yz)\}^2}\rho_{yz}^2-\frac{E(yz^3)}{E(z^2)E(yz)}\rho_{yz}^2$$

$$+\frac{E(x^2yz)}{E(x^2)E(yz)}\rho_{xy}\rho_{yz}+\frac{E(xyz^2)}{E(z^2)E(xy)}\rho_{xy}\rho_{yz}-2\frac{E(xy^2z)}{E(xy)E(yz)}\rho_{xy}\rho_{yz}$$

$$+O\left(\frac{1}{n^2}\right)$$

区を単位とした時の昭和２２年の焼失坪数と昭和２３年の焼失坪数の相関係数は０.４６０３で，昭和２３年と２４年では０.３６７３である。

今焼失坪数とゆうものは社会経済的な状態とか，気象等の他に random factor によつて作用されると考える。例えば，昭和２２年に於ける千代田区の焼失坪数１４１８坪とゆうものは昭和２２年の千代田区とゆうものに対して random factor によつて無限に変動する量の一つの実現値であると考えるのである。ここでこの分布は我々にとつて知り得ないものであるから，これを各区の実現値が示す分布でおきかえる。



X　は実現値
A　の一つ一つの分布を，Bの分布でおきかえるのである.

ちなみに，各区について，昭和２２年，２３年，２４年の焼失坪数の variance を求めて平均してみると，463929である。又，昭和２２年，２３年，２４年の各年について，区の焼失坪数の varianceは，467004，840529，464556であり，variance の order は一致している。

これからみても上の formulation がそれほどでたらめでもないであらう。

この様に formulate すれば２３区の値は random factor によつて変動する量の２３個の sampleと考えられるから，前に計算した $\sigma^2_{(r_{xy}-r_{yz})}$ を用いて，相関係数の有意差の検定を行ふ。　実際に計算に行えば

$$\sigma^2_{(r_{xy}-r_{yz})} = 0.072664 \quad \text{であり，} \quad \sigma = 0.269$$

であるから　$$\frac{0.4603-0.3673}{0.269} = 0.345$$　となり

0.4603　と　0.3673　の間には有意な差はない。

つまり，　年によつて相関係数は　0.4603と0.3673とちがつてはいるけれど，この２つは random factor によつて変動する範囲内にあるのであり，年によつて実質的な差はないのである。

When sample of size $\underline{n}$ is taken from infinite population whose elements have three labels, X, Y, Z, the expectation and variance of the difference between two sample correlation coefficients r_{xy}, r_{yz}, are as follows :

$$E(r_{xy}) = \rho_{xy}\Big\{1-\frac{1}{n}\Big(\frac{1}{2}\cdot\frac{E(x^3y)}{E(x^2)E(xy)}+\frac{1}{2}\frac{E(xy^3)}{E(y^2)E(xy)}$$

$$-\frac{1}{4}\frac{E(x^2y^2)}{E(x^2)E(y^2)}-\frac{3}{8}\frac{E(x^4)}{(E(x^2))^2}-\frac{3}{8}\frac{E(y^4)}{\{E(y^2)\}^2}+O\Big(\frac{1}{n^2}\Big)\Big\}\ ,$$

$$\sigma^2(r_{xy}-r_{yz}) = \frac{1}{n}\frac{E(y^4)}{\{E(y^2)\}^2}(\rho_{xy}-\rho_{yz})^2+\cdot\frac{E(x^2y^2)}{E(x^2)E(y^2)}(\rho_{xy}^2-\rho_{xy}\rho_{yz})$$

$$-\frac{E(xy^3)}{E(y^2)E(xy)}(\rho_{xy}^2-\rho_{xy}\rho_{yz})+\frac{1}{2}\cdot\frac{E(y^2z^2)}{E(y^2)E(z^2)}(\rho_{yz}^2-\rho_{xy}\rho_{yz})$$

$$-\frac{E(y^3z)}{E(y^2)E(yz)}(\rho_{yz}^2-\rho_{xy}\rho_{yz})+\frac{1}{4}\frac{E(x^4)}{\{E(x^2)\}^2}\rho_{xy}^2$$

$$+\frac{1}{4}\frac{E(z^4)}{\{E(z^2)\}^2}\rho_{yz}^2-\frac{1}{2}\frac{E(x^2z^2)}{E(x^2)E(z^2)}\rho_{xy}\rho_{yz}+\frac{E(x^2y^2)}{\{E(xy)\}^2}\rho_{xy}^2$$

$$-\frac{E(x^3y)}{E(x^2)E(xy)}\rho_{xy}^2+\frac{E(y^2z^2)}{\{E(yz)\}^2}\rho_{yz}^2-\frac{E(yz^3)}{E(z^2)E(yz)}\rho_{yz}^2$$

$$+\frac{E(x^2yz)}{E(x^2)E(yz)}\rho_{xy}\rho_{yz}+\frac{E(xyz^2)}{E(z^2)E(xy)}\rho_{xy}\rho_{yz}-2\frac{E(xy^2z)}{E(xy)E(yz)}\rho_{xy}\rho_{yz}\Big\}$$

$$+\ O\Big(\frac{1}{n^2}\Big)\ ,$$

For example, the burned-down area is often influenced by random factor as well as various socio-economical states, and weather.

The burned down area of Chiyoda-ku in 1947 is a sample value from distribution of the burned-down areacaused by random factor.

This distribution, being unknown, is replaced by that of sample value of the aggregated burned-down area.

The latter sample value is considered as that obtained from the distribution mentioned aboved.

Therefore, expectation and variance we have refferred to may be used to test the hypothesis that two correlation coefficient are equal.

(1951. 9.5 受付)