and Theorem I., page two), Craig's 1938 paper on the subject was an expository paper, presented to the Institute at the invitation of the Program Committee. He had no part in the discovery of the theorem.

22 ON THE INDEPENDENCE OF STATISTICS OF QUADRATIC FORMS

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In 1934, W. G. Cochran[1] gave a criterion of independency of statistics of quadratic form, formed from a sample of size N taken independently from a univariate normal population, as the rankrelation of their coefficient matrices. This is the so-called "Cochran's Theorem".[2] Later, in 1938, A. T. Crig stated the same theorem and proved it algebraically. But his proof was very tedious. In practical use, however, it is not always easy to determine the rank of a matrix, so

H. Sakamoto[4] gave the independency criterion of two statistics of quadratic forms as the orthogonality relation between their coefficient matrices. In this paper I shall extend the Cochran-Craig's theorem somewhat, and show that these conditions are equivalent to that of Sakamoto, and furthermore, by the use of results of recent linear algebra, I shall try to simplify the proofs from the standpoint of uniformity.

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(1) W. G. Cochran; The distribution of quadratic forms in a normal system, with applications to the analysis of covariance. Proceedings of the Cambridge Philosophical Society, Vol. 30, 1934.

(2) S. S. Wilks; Mathematical Statistics. 1943, P. 107.

(3) A. T. Craig; On the independence of certain estimate of variance Annals of Mathematical Statistics, Vol. 9, 1938.

(4) Heihachi Sakamoto; On the independence of statistics (Tokeiryo no Dekuristusei ni tsuite). Research Memoir of the Institute of Statistical Mathematics, Vol. I, NO. 9.

About the symbols and techniques used in the following, the writer refers to Sakamoto's paper quoted above and to the following two books:

Sperner-Schreior, Vorlesungen über die Theorie der Matrizen, 1932. Sperner-Schreier, Einführung in die Analytische Geometrie und Algebra, Bd. II, 1935.

§ I. THEOREMS.

THEOREM I. (The extension of Cochran-Craig Theorem)[1]

A vector $\mathfrak{X}=(x_1, x_2, \cdots, x_n)$ being subject to the joint distribution law.

$$f(\mathfrak{X})d\mathfrak{X}=(2\pi)^{-\frac{1}{2}n}|V|^{\frac{1}{2}}\exp.\left(-\tfrac{1}{2}(V^{-1}\mathfrak{X}\cdot\mathfrak{X})\right)d\mathfrak{X}.$$

Where the matrix V being the variance matrix, and $d\mathfrak{X}=dx_1 \cdots dx_n$.

Then the necessary and sufficient condition, that s statistics $\theta_i=(A_i\mathfrak{X}\cdot\mathfrak{X})$, $i=1.2.\cdots\cdots; s$ are mutually independent, is the following rankrelation helds; i.e.

$$\sum_{i=1}^{s} r_i = R.$$

where r_i, $i=1.2.\cdots\cdots, s$ be the rank of A_i, $i=1.2.\cdots\cdots, s$ and R be the rank of the matrix

$$B=\sum_{i=1}^{s} A_i$$

THEOREM II. (Sakamoto's Theorem)[2]

Under the same conditions as those of Theorem I, the relation $\sum_{i=1}^{s} r_i = R$ is equivalent to the following, that is, the necessary and sufficient conditions that s statistics θ_i, $i=1,2,\cdots\cdots,S$ are mutually independent, is the following relations held;

$$A_i V A_j = 0, \text{ for all } i \neq j.$$

THEOREM III. Under the same conditions as those of Theorem I, it follows from the relation $BVB = B$ the following s relations

$$A_i V A_i = A_i \quad i = 1, 2, \cdots\cdots S.$$

THEOREM IV.[3] Under the same conditions as those of Theorem I, the necessary and sufficient condition that the statistic $\theta = (A\mathcal{Y}\cdot\mathcal{Y})$ is subject to χ^2-distribution of degrees of freedom f is that

$$A V A = A$$

and the rank of A is f.

THEOREM V.[4] Under the same conditions as those of Theorem I, when the statistic $\theta = (B\mathcal{Y}\cdot\mathcal{Y})$ is subject to χ^2-distribution of degrees of freedom R, then each of $\theta_i = (A_i\mathcal{Y}\cdot\mathcal{Y})$ is subject to χ^2-distribution of degrees of freedom r_i

(1) W. G. Cochran; Loc. cit. A. T. Craig; Loc. cit
S. S. Wilks; Loc. Loc.
(2) H. Sakamoto; Loc. cit.
Motasaburō Masuyama; Shōsūei no matemokata to jikkenkeigaku no tatekata (2nd edition) P. 114.
(3) H. Sakamoto; Loc. cit.
(4) H. Sakamoto; Loc. cit.

§ 2. LEMMATA

To prove the theorems stated in § 1, we shall have to prove some Lommata.

LEMMA I. Under the same conditions as those of Theorem I, the characteristic function $\varphi(t)$ of a statistic $\theta=(A\xi\cdot\xi)$ is written in the form.

$$\varphi(t)=|E-2itA^{*}|^{-\frac{1}{2}}$$

where A^{*} is a symmetric matrix, which is transformed from A by a certain matrix.

PROOF:

$$\varphi(t)=\int_{R(\xi)} e^{it\theta} f(\xi)\,d\xi$$

$$=(2\pi)^{-\frac{1}{2}n}|V|^{-\frac{1}{2}}\int_{R(\xi)} e^{it(A\xi\cdot\xi)-\frac{1}{2}(V^{-1}\xi\cdot\xi)} \cdot a$$

$(V^{-1}\xi\cdot\xi)$ is a pastive definite quadratic form, therefore, by taking a suitable orthogonal matrix T, we can transform V^{-1} into the diagonalform:

$$T^{-1}V^{-1}T_{\circ}=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix},\ \lambda_i>0.\ (i=1,2,\cdots\cdots,n)$$

Since the Jacobian of this transformation is unity, putting $T^{-1}AT=\tilde{A}$, we have

$$\varphi(t)=(2\pi)^{-\frac{1}{2}n}|V|^{-\frac{1}{2}}\int e^{-\frac{1}{2}(\lambda_1 x_1^2+\cdots\cdots+\lambda_n x_n^2-2it(\tilde{A}\mathfrak{x}\cdot\mathfrak{x}))}d\mathfrak{x}$$

Then transforming by the equation

$$\sqrt{\lambda_i}\cdot x_i=y_i,\ i=1,2,\cdots\cdots,n,\ \text{i.e.}\ \mathfrak{y}=\begin{pmatrix}\sqrt{\lambda_1} & & 0\\ & \ddots & \\ & & \sqrt{\lambda_n}\end{pmatrix}\mathfrak{x}=\Lambda^{-1}\mathfrak{x},$$

the variable $\mathfrak{x}$ into $\mathfrak{y}$. Jacobian of this transformation being $|\sqrt{\lambda_1\cdots\lambda_n}|$, we have

$$\varphi(t)=(2\pi)^{-\frac{1}{2}n}|V|^{-\frac{1}{2}}|\lambda_1\cdots\lambda_n|^{-\frac{1}{2}}\int_{R(\mathfrak{y})}e^{-\frac{1}{2}((\mathfrak{y}\mathfrak{y})-2it(\Lambda'\tilde{A}\Lambda\,\mathfrak{y}\mathfrak{y}))}$$

where $|V^{-1}|=\lambda_1\cdots\cdots\lambda_n$.

Putting $A^{*}=\Lambda'\tilde{A}\Lambda$, we have the following equation:

$$\varphi(t)=(2\pi)^{-\frac{1}{2}n}\int_{R(\mathfrak{y})}e^{-\frac{1}{2}((\mathfrak{y}\mathfrak{y})-2it(A^{*}\mathfrak{y}\mathfrak{y}))}d\mathfrak{y}.$$

Since the matrix A^{*} is a symmetric one, if we transform it by a suitable orthogonal matrix S, i.e., putting $\mathfrak{y}=S\mathfrak{z}$, we have

$$S^{-1}A^{*}S=\begin{pmatrix}\sigma_1 & & 0\\ & \ddots & \\ 0 & & \sigma_n\end{pmatrix}$$

So we have

$$\varphi(t)=(2\pi)^{-\frac{1}{2}n}\int_{R(\mathfrak{z})} e-\tfrac{1}{2}\{(\mathfrak{z}\mathfrak{z})-2it(S^{-1}A^{*}S\mathfrak{z}\mathfrak{z})\}$$

$$=(2\pi)^{-\frac{1}{2}n}\prod_{j=1}^{n}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(1-2it\sigma_j)Z_j^2}dZ_j.$$

Setting

$$\sqrt{1-2it\sigma_j}\cdot Z_j=\zeta_j,\quad j=1,\dots,n,$$

We have

$$\varphi(t)=(2\pi)^{-\frac{1}{2}n}\left[\prod_{j=1}^{n}(1-2it\sigma_j)\right]^{-\frac{1}{2}}\prod_{j=1}^{n}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\zeta_j^{2}}d\zeta_j$$

$$=\prod_{j=1}^{n}(1-2it\sigma_j)^{-\frac{1}{2}}$$

$$=|E-2itS^{-1}A^{*}S|^{-\frac{1}{2}}=|E-2itA^{*}|^{-\frac{1}{2}}$$

Q. E. D.

LEMMA 2. The necessary and sufficient condition that s statistics $\theta_i=(A_i\mathfrak{x}\cdot\mathfrak{x})$, $i=1,2,\dots\dots,s$ are independent mutually, is that the relation between their characteristic functions holds;

$$\varphi(t_1,\cdots,t_s)=\varphi(t_1)\cdots\cdots\varphi(t_s),$$

here $\varphi(t_i)$ is the characteristic function corresponding to θ_i and $\varphi(t_1,\cdots,t_s)$ the one corresponding to $\theta=\sum_{i=1}^{S}\theta_i$.

LEMMA 3. From the LEMMA 1 and 2 stated above, we can easily see that the necessary and sufficient condition that S statistics

$$\theta_j=(A_j\mathfrak{X}\cdot\mathfrak{X}),\quad j=1,2,\cdots\cdots,S$$

are mutually independent; is that for all real values of $t_1, t_2, \cdots\cdots t_s$ the following relation holds

$$|E-2it_1A_1^*\cdots\cdots-2it_sA_s^*|$$

$$=\prod_{j=1}^{S}|E-2it_jA_j^*|$$

LEMMA 4. $A_i\ (i=1,2,\cdots\cdots,S)$ being S symmetric matrices of degree n, and let $\sum_{i=1}^{S}A_i=B$. If for all real values of t, the relation

$$|E-tB|=\prod_{i=1}^{S}|E-tA_i|$$

holds, then the rank-relation,

$$R = r_1 + \cdots\cdots + r_s$$

holds, where R and r_i $(i = 1, 2, \cdots, s)$. being the rank of B and A_i $(i = 1, 2, \cdots, s)$ respectively.

PROOF: Letting $t = \frac{1}{x}$, then we have

$$x^{(s-1)n} |xE - B| = \prod_{i=1}^{s} |xE - A_i|$$

Equating the numbers of zero-factors of both sides, we have

$$(s-1)n + b = \sum_{i=1}^{s} a_i,$$

Where a_i and b being the number of zero-factors of the determinant $|XE - A_i|$ and $|XE - B|$ respectively.

Hence we have

$$sn + b = n + \sum_{i=1}^{s} a_i.$$

Therefore,

$$\sum_{i=1}^{s} (n - a_i) = (n - b)$$

Now, by the writer's note: "on the algebraical proof of Mr. Sakamoto's Lemma", when we think a matrix of degree n as a linear transformation in a linear vector space of dimension n, and when the matrix is symmetric, then the dimension of its zero-point set[2] is equal to the number of zero-factors in its characteristic polynomial. Therefore the rank of a symmertric matrix is eqical to the difference of n and the dimension of its zero-point set. Hence we can write the above relation in the form;

$$R = r_1 + \cdots\cdots + r_s$$

Q. E. D.

LEMMA 5. The ranks of S matrices A_i $(i = 1, 2, \cdots, S)$ Be r_i $(i = 1, 2, \cdots S)$ respectively, and the rank of the matrix $B = \sum_{i=1}^{S} A_i$ be R. When

$$\sum_{i=1}^{S} r_i = R$$

and furthermore, the matrix B is idempetent, then

$$A_i^2 = A_i, \quad \text{and} \quad A_i A_j = 0 \quad \text{for all } i \neq j.$$

PROOF: We shall prove this Lemma by the method of mathematical induction.

(1) Proof for the orthogenality relations.

The case when $S = 2$.

Let us see three symmetric A_1, A_2 and $B = A_1 + A_2$ as Linear transformations in n-dimensional Vector space $\mathcal{L}$ over the field of real numbers. And let the zero-point sets of A_1, A_2 and B be $\mathfrak{N}_1$, $\mathfrak{N}_2$ and $\mathfrak{N}$ respectively, then the following three direct sum decompositions are taken place.[3]

$$\mathfrak{L} = \mathfrak{M}_1 + \mathfrak{N}_1 = \mathfrak{M}_2 + \mathfrak{N}_2 = \mathfrak{M} + \mathfrak{N},$$

Where $\mathfrak{N}_1, \mathfrak{N}_2$ and $\mathfrak{N}$ are subspaces of $\mathfrak{L}$ containing all the vectors of $\mathfrak{L}$ which are mapped on zero-vectors by A_1, A_2 and B respectively, and they, of course, form lenear subspaces of $\mathfrak{L}$. By the writer's note[4] quoted above, the dimensions of $\mathfrak{M}_1, \mathfrak{M}_2$ and $\mathfrak{M}$ are r_1, r_2 and R respectively, therefore the dimensions of $\mathfrak{N}_1, \mathfrak{N}_2$ and $\mathfrak{N}$ are $n-r_1$, $n-r_2$ and $n-R$ respectively. If a vector $\mathfrak{x}$ is contained in $\mathfrak{N}_1 \wedge \mathfrak{N}_2$, then from the definition of zero-point sets.

$$A_1\mathfrak{x} = 0, \quad A_2\mathfrak{x} = 0.$$

Therefore, we have

$$B\mathfrak{x} = A_1\mathfrak{x} + A_2\mathfrak{x} = 0,$$

that is

$$\mathfrak{x} \in \mathfrak{N}$$

In other words,

$$\mathfrak{N}_1 \wedge \mathfrak{N}_2 \leq \mathfrak{N}.$$

Therefore

$$\dim(\mathfrak{N}_1 \wedge \mathfrak{N}_2) \leq \dim(\mathfrak{N}).$$

Now, from the well-known fact that[5]

$$\dim(\mathfrak{N}_1 \cdot \mathfrak{N}_2) \leq \dim(\mathfrak{N}_1) + \dim(\mathfrak{N}_2) - \dim(\text{[illegible]}$$

We have the following inequality:

$$\dim(\mathfrak{N}_1 \cdot \mathfrak{N}_2) \geq \dim(\mathfrak{N}_1) + \dim(\mathfrak{N}_2) - \dim(\text{[illegible]}$$

This is

$$n \geq \dim(\mathfrak{N}_1, \mathfrak{N}_2) \geq (n - r_1) + (n - r_2) - (n - R)$$

From the hypethesis of this Lamma, we have

$$\dim(\mathfrak{N}_1 \cdot \mathfrak{N}_2) \geq n.$$

Since $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are two subspaces of $\mathcal{L}$, their join is also a subspace of $\mathcal{L}$. Therefore, from the inequality obtained above, we have

$$\mathcal{L} = (\mathfrak{N}_1 \circ \mathfrak{N}_2).$$

Therefore

$$\mathfrak{M}_1 \wedge \mathfrak{M}_2 = 0$$

This represents the orthogonality between A_1 and A_2.

(II) The general case.

In general case, we prove by mathematical induction. We assume that the theorem holds for $s-1$ symmetric matrices, and then prove the theorem for s symmetric matrices. For example, without loss of generality, for $A_1, A_2, \cdots\cdots, A_{s-1}$ we assume that the theorem helds: Putting

$$B' = A_1 + \cdots + A_{s-1},$$

We have the rank R' of B' is equal to the sum

$$r_1 + \cdots\cdots + r_{s-1}$$

For the rank R' is alearly not greater than the sum $r_1 + \cdots + r_{s-1}$, and by means of the same reasoning applied to the relation $B = B' + A_s$, we have the relation

$$R \leq R' + r_s.$$

Hence $$R = R' + r_s.$$

Therefore, from the proof in the case $s=2$, we see that

$$B' A_s = 0$$

This is $$(A_1 + A_2 + \cdots\cdots + A_{s-1}) A_s = 0,$$

$$A_1 A_s + \cdots\cdots + A_{s-1} A_s = 0.$$

Multiplying A_i $(1 \leq i \leq s-1)$ from the left we have, from the assumption of induction that

$$A_i^2 A_{s-1} = 0.$$

Letting $A_i A_{s-1} = C_{i,s-1}$, since A_i $(1 \leq i \leq s)$ is symmetric

$$C'_{i,s-1} = A_{s-1} A_i.$$

Therefore $C'_{i,s-1} \cdot C_{i,s-1} = A_{s-1} \cdot A_i \cdot A_i \cdot A_{s-1}$

$$= A_{s-1} \; A_i^2 A_{s-1} = 0$$

Therefore, we have

$$C_{i,s-1} = 0.$$

That is

$$A_i A_s = 0 \quad \text{for all } 1 \leq i \leq s-1.$$

(II) Proof for idempetency

(i) The case when $s=2$.

If B is idempetent, because of the orthogonality between A_1 and A_2 for any vector $\mathscr{C}$ in $\mathscr{L}$, we have the following relation;

$$A_1^2 \mathscr{C} + A_2^2 \mathscr{C} = A_1 \mathscr{C} + A_2 \mathscr{C}.$$

Therefore,

$$A_1(A_1\mathfrak{X}-\mathfrak{X}) = A_2(-A_2\mathfrak{X}+\mathfrak{X}).$$

Now, from the fact that $\mathfrak{M}_1 \frown \mathfrak{M}_2 = 0$, it follows that both sides of above equation must vanish. That is

$$A_1^2\mathfrak{X} = A_1\mathfrak{X}, \quad A_2^2\mathfrak{X} = A_2\mathfrak{X}.$$

for any vector $\mathfrak{X}$ in $\mathfrak{L}$.

In other words, A_1 and A_2 are idempetent.

(ii) The general case.

We proceed by the mathematical induction. Without loss of generality, assuming that the theorem holds for the first $s-1$ symmetric matrices $A_1, A_2, \cdots, A_{s-1}$.

Putting

$$B' = A_1 + \cdots\cdots + A_{s-1},$$

We have

$$B = B' + A_s.$$

If B is idempetent, then from the proof above, we have B' and A_s are idempetent. By the assumption of induction, it follows that A_i $(1 \leq i \leq s)$ is idempetent.

Thus the proof of LEMMA 5 is completed.

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(1) Junjiro Ogawa; On the algebraical proof of Mr. Sakamoto's Lemm Research Memoir of the Institute of Statistical Mathematics,

Vol. 1 NO. 15.

(2) Sperner-Schreier; Verlesungen Uber Matrizen, P. 47-.
(3) Sperner-Schreier; Loc. cit.
(4) J. Ogawa; Loc. cit.
(5) Sperner-Schreier; Einfuhrung in die Analytische Geometrie und Algebra Bd. I, P. 28, Sätz 7.

§ 3. PROOFS OF THEOREMS

From Lemmata of § 2, the proofs of theorems are straightforward.

The writer refers to the Research Memoir of the Institute of Statistical Mathematics, Vol. 11, NO. 4: Junjiro Ogawa; On the independen e of statistics of quadratic forms.

(8. March, 1946, in Tokyo.)