Algorithmic analogies to Kamae-Weiss theorem on normal numbers

1 Selection function

Example

\[
\begin{align*}
  x &= 0 \ 1 \ 0 \ 0 \ 1 \ \cdots \\
  y &= 1 \ 0 \ 1 \ 0 \ 1 \ \cdots \\
  x/y &= 0 \ 0 \ 1 \ 0 \ 0 \ \cdots 
\end{align*}
\]

\(x, y \in \{0, 1\}^\infty\).

2 Kamae-Weiss theorem

**Definition 1.** \(x = x_1x_2 \cdots\) is called normal number if

\[
\forall s \in \{0, 1\}^* \lim_{n \to \infty} \#\{1 \leq i \leq n \mid x_i \cdots x_{i+|s|-1} = s\} / n = 2^{-|s|}.
\]

Let \(\mathcal{N}\) be the set of normal numbers.

**Definition 2.** \(p\) is called cluster point if there is a sequence \(\{n_i\}\)

\[
\forall s p(s) = \lim_{i \to \infty} \#\{1 \leq j \leq n_i \mid x_j \cdots x_{j+|s|-1} = s\} / n_i.
\]

Let \(V(x)\) be the set of cluster point of \(x\).

\(V(x) \neq \emptyset\) for all \(x\).

Kamae entropy is defined by

\[
h(x) = \sup\{h(p) \mid p \in V(x)\}.
\]

**Theorem 1** (Kamae[1]). Suppose that \(\lim \inf \frac{1}{n} \sum_{i=1}^{n} y_i > 0\) then (i) and (ii) are equivalent:

(i) \(h(y) = 0\).

(ii) \(\forall x \in \mathcal{N} \ x/y \in \mathcal{N}\).

Note: The part (i) \(\Rightarrow\) (ii) is appeared in Weiss [3].

3 van Lambalgen’s conjecture

\(K\) : prefix Kolmogorov complexity.

\(\mathcal{R}\) : the set of Martin-Löf random sequences with respect to \((1/2,1/2)\)-i.i.d. process.

In van Lambalgen [2] the following equivalence is conjectured:

(i) \(\lim_{n \to \infty} K(y^n_1)/n = 0\).

(ii) \(\forall x \in \mathcal{R} \ x/y \in \mathcal{R}\).

4 Proposition 1

**Proposition 1.** Suppose that \(y\) is Martin-Löf random with respect to some computable probability \(P\) and \(\sum_{i=1}^{\infty} y_i = \infty\). Then the following two statements are equivalent:

(i) \(y\) is computable.

(ii) \(\forall x \in \mathcal{R} \ x/y \in \mathcal{R}\).

\(\mathcal{R}\) : the set of Martin-Löf random sequences with respect to \((1/2,1/2)\)-i.i.d. process relative to \(y\).

5 Weak randomness

**Definition 3.** \(y\) is called weakly random with respect to a computable \(P\) if

\[
\lim_{n \to \infty} K(y^n_1)/n = \lim_{n \to \infty} \frac{1}{n} \log P(y^n_1).
\]

\(y\) is weakly random with respect to \((1/2,1/2)\)-i.i.d. process if

\[
\lim_{n \to \infty} K(y^n_1)/n = 1.
\]

Note

\(y \in \mathcal{R} \rightarrow \lim_{n \to \infty} K(y^n_1)/n = 1 \rightarrow y \in \mathcal{N}\).

None of the converse is true.

6 Proposition 2

**Proposition 2.** Suppose that \(y\) is weakly random with respect to a computable measure and \(\lim \frac{1}{n} \sum_{i=1}^{n} y_i > 0\).

Then the following two statements are equivalent:

(i) \(\lim_{n \to \infty} K(x^n_1)/n = 0\).

(ii) \(\forall x \lim_{n \to \infty} K(x^n_1)/n = 1 \rightarrow \lim_{n \to \infty} \frac{1}{x^n_1/y^n_1} K(x^n_1/y^n_1) = 1\).

Example: computable sequences and sturmian sequences satisfy (i).

References

