Dependence Structure of Bivariate Order Statistics and its Applications

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Abstract
We study the dependence structure of bivariate order statistics, and prove that if the underlying bivariate distribution \( H \) is positive quadrant dependent (PQD) then so is each pair of bivariate order statistics. As an application, we show that if \( H \) is PQD, the bivariate distribution \( K_{n}^{(n)} \), proposed by Bairamov and Bayramoglu (2012), is greater than or equal to Baker’s (2008) distribution \( H_{n}^{(n)} \). We also show that if \( H \) is PQD, \( K_{n}^{(n)} \) converges weakly to the Fréchet–Hoeffding upper bound as \( n \) tends to infinity.

Introduction
Bivariate order statistics
Suppose that \( (X_{1}, Y_{1}), \ldots, (X_{n}, Y_{n}) \sim i.i.d. \) \( H(x, y) = \Pr(X \leq x, Y \leq y) \).
Margins: \( F(x) = \Pr(X \leq x) \); \( G(y) = \Pr(Y \leq y) \)
Order statistics: \( X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} \); \( Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n} \)

Distribution functions:
\[
F_{n}(x, y) := \Pr(X_{n} \leq x, Y_{n} \leq y) = \Pr(X_{n} \leq x) \Pr(Y_{n} \leq y) = F(x) \cdot G(y).
\]

Positive quadrant dependence (PQD): \( H(x, y) \geq F(x)G(y) \) for all \( x, y \).
Negative quadrant dependence (NQD): \( H(x, y) \leq F(x)G(y) \) for all \( x, y \).

Dependence Structure
Theorem 1.
For \( 1 \leq r, s \leq n \), the distribution \( K_{n}^{(r,s)} \) is increasing in \( H \).
Proof: \( \frac{\partial}{\partial H} K_{n}^{(r,s)}(x, y) = n f_{n}^{(r-1,s-1)}(x, y) \geq 0 \).

References
- Bairamov, I. and Bayramoglu, K. (2013). From the Huang–Kotz FGM distribution to Baker’s (2008) distribution \( H_{n}^{(n)} \). We also show that if \( H \) is PQD, \( K_{n}^{(n)} \) converges weakly to the Fréchet–Hoeffding upper bound as \( n \) tends to infinity.

Corollary 1.
For \( 1 \leq r \leq s \leq n \), the joint distribution of \( (X_{r:n}, X_{s:n}) \), \( K_{n}^{(r,s)} \), is PQD if \( H \) is PQD, and is NQD if \( H \) is NQD.

Theoretical Applications
Baker’s (2008) distribution:
\[
H_{n}^{(n)}(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(X_{i} \leq x, Y_{j} \leq y), \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(X_{i} \leq x, Y_{j} \leq y) = \frac{1}{n} \cdot r_{w} \geq 0.
\]

Bairamov and Bayramoglu’s (2013) distribution:
\[
K_{n}^{(n)}(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(X_{i} \leq x, Y_{j} \leq y), \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(X_{i} \leq x, Y_{j} \leq y) = \frac{1}{n} \cdot r_{w} \geq 0.
\]

Monotonicity of \( K_{n}^{(n)}(x, y) \)
Fact: As \( n \to \infty \), \( H_{n}^{(n)}(x, y) \to \min\{F(x), G(y)\} \) (Dou et al. 2013)
Problem: As \( n \to \infty \), \( K_{n}^{(n)}(x, y) \to \min\{F(x), G(y)\} \) monotonically increases in \( n \)?

Theorem 2.
(i) \( K_{n}^{(n)} \geq H_{n}^{(n)} \) if \( K_{n}^{(n)} \geq H_{n}^{(n)} \) depending on \( H \) is PQD or NQD.
(ii) \( K_{n}^{(n)} \geq H_{n}^{(n)} \) if \( K_{n}^{(n)} \geq H_{n}^{(n)} \) depending on \( H \) is PQD or NQD.

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